# Causal Interaction in High Dimension

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# Interaction Effects and Causal Heterogeneity

1. Moderation
   - How do treatment effects vary across individuals?
   - Who benefits from (or is harmed by) the treatment?
   - Interaction between treatment and pre-treatment covariates

2. Causal interaction
   - What aspects of a treatment are responsible for causal effects?
   - What combination of treatments is efficacious?
   - Interaction between treatment variables

3. Individualized treatment regimes
   - What combination of treatments is optimal for a given individual?

# Causal Interaction in High Dimension

- High dimension = many treatments, each having multiple levels
- A motivating application: Conjoint analysis (Hainmueller *et al.* 2014)
  - survey experiments to measure immigration preferences
  - a representative sample of 1,396 American adults
  - each respondent evaluates 5 pairs of immigirant profiles
  - `gender`$^{2}$, `education`$^{7}$, `origin`$^{10}$, `experience`$^{4}$, `plan`$^{4}$, `language`$^{4}$, `profession`$^{11}$, `application reason`$^{3}$, `prior trips`$^{5}$
  - Over 1 **million** treatment combinations!
  - What combinations of immigrant characteristics make them preferred?
- Too many treatment combinations $\rightsquigarrow$ Need for an effective summary
- Interaction effects play an essential role

# Two Interpretations of Causal Interaction

1. Conditional effect interpretation:
   - Does the effect of one treatment change as we vary the value of another treatment?
   - Does the effect of being black change depending on whether an applicant is male or female?
   - Useful for testing moderation among treatments

2. Interactive effect interpretation:
   - Does a combination of treatments induce an *additional effect* beyond the sum of separate effects attributable to each treatment?
   - Does being a black female induce an additional effect beyond the effect of being black and that of being female?
   - Useful for finding efficacious treatment combinations in high dimension

# An Illustration in the $2 \times 2$ Case

- Two binary treatments: $A$ and $B$
- Potential outcomes: $Y(a, b)$ where $a, b \in \{0, 1\}$
- Conditional effect interpretation:

$$\underbrace{[Y(1,1) - Y(0,1)]}_{\text{effect of } A \text{ when } B=1} - \underbrace{[Y(1,0) - Y(0,0)]}_{\text{effect of } A \text{ when } B=0}$$

$\rightsquigarrow$ requires the specification of moderator

- Interactive effect interpretation:

$$\underbrace{[Y(1,1) - Y(0,0)]}_{\text{effect of } A \text{ and } B} - \underbrace{[Y(1,0) - Y(0,0)]}_{\text{effect of } A \text{ when } B=0} - \underbrace{[Y(0,1) - Y(0,0)]}_{\text{effect of } B \text{ when } A=0}$$

$\rightsquigarrow$ requires the specification of baseline condition

- The same quantity but two different interpretations

# Difficulty of the Conventional Approach

- Lack of invariance to the baseline condition
  $\rightsquigarrow$ Inference depends on the choice of baseline condition
- $3 \times 3$ example:
  - Treatment $A \in \{a_0, a_1, a_2\}$ and Treatment $B \in \{b_0, b_1, b_2\}$
  - Regression model with the baseline condition $(a_0, b_0)$:

$$\mathbb{E}(Y \mid A, B) \quad = \quad 1 + a_1^* + a_2^* + b_2^* + a_1^* b_2^* + 2a_2^* b_2^* + 3a_2^* b_1^*$$

  - Interaction effect for $(a_2, b_2)$ $>$ Interaction effect for $(a_1, b_2)$
  - Another equivalent model with the baseline condition $(a_0, b_1)$:

$$\mathbb{E}(Y \mid A, B) \quad = \quad 1 + a_1^* + 4a_2^* + b_2^* + a_1^* b_2^* - a_2^* b_2^* - 3a_2^* b_0^*$$

  - Interaction effect for $(a_2, b_2)$ $<$ Interaction effect for $(a_1, b_2)$
  - Interaction effect for $(a_2, b_1)$ is zero under the second model
  - All interaction effects with at least one baseline value are zero

# Empirical Illustration: Lack of Invariance

- Linear regression with main effects and two-way interactions
- Baseline: *lowest* levels of job experiences and education

| | Education | | | | | | |
|---|---|---|---|---|---|---|---|
| **Job experience** | None | 4th grade | 8th grade | High school | Two-year college | College | Graduate |
| None | 0 (baseline) | 0 | 0 | 0 | 0 | 0 | 0 |
| 1–2 years | 0 | 0.009 (0.063) | −0.019 (0.063) | −0.032 (0.063) | 0.100 (0.064) | −0.044 (0.064) | −0.064 (0.063) |
| 3–5 years | 0 | 0.016 (0.063) | 0.056 (0.064) | 0.165 (0.064) | 0.107 (0.064) | 0.010 (0.065) | 0.117 (0.063) |
| $>5$ years | 0 | −0.050 (0.064) | 0.126 (0.064) | 0.042 (0.063) | 0.058 (0.064) | −0.094 (0.064) | 0.015 (0.064) |

# The Effects of Changing the Baseline Condition

- Same linear regression but different baseline
- Baseline: *highest* levels of job experiences and education

| **Job experience** | **Education** None | 4th grade | 8th grade | High school | Two-year college | College | Graduate |
|---|---|---|---|---|---|---|---|
| None | 0.015 (0.064) | 0.065 (0.062) | −0.111 (0.064) | −0.027 (0.061) | −0.043 (0.063) | 0.109 (0.063) | 0 |
| 1–2 years | 0.078 (0.064) | 0.138 (0.062) | −0.066 (0.062) | 0.006 (0.061) | 0.120 (0.062) | 0.129 (0.062) | 0 |
| 3–5 years | −0.102 (0.062) | −0.036 (0.062) | −0.172 (0.063) | 0.021 (0.062) | −0.054 (0.061) | 0.002 (0.062) | 0 |
| $>$ 5 years | 0 | 0 | 0 | 0 | 0 | 0 | 0 (baseline) |

# The Contributions of the Paper

1. Problems of the conventional approach:
   - Lack of invariance to the choice of baseline condition
   - Difficulty of interpretation for higher-order interaction
2. Solution: Average Marginal Treatment Interaction Effect
   - invariant to baseline condition
   - same, intuitive interpretation even for high dimension
   - simple estimation procedure
3. Reanalysis of the immigration survey experiment

# Two-way Causal Interaction

- Two factorial treatments:

$$\begin{aligned} A &\in \mathcal{A} = \{a_0, a_1, \ldots, a_{D_A-1}\} \\ B &\in \mathcal{B} = \{b_0, b_1, \ldots, b_{D_B-1}\} \end{aligned}$$

- Assumption: Full factorial design
  1. Randomization of treatment assignment

$$\{Y(a_\ell, b_m)\}_{a_\ell \in \mathcal{A}, b_m \in \mathcal{B}} \perp\!\!\!\perp \{A, B\}$$

  2. Non-zero probability for all treatment combination

$$\Pr(A = a_\ell, B = b_m) > 0 \quad \text{for all } a_\ell \in \mathcal{A} \quad \text{and} \quad b_m \in \mathcal{B}$$

- Fractional factorial design not allowed
  1. Use a small non-zero assignment probability
  2. Focus on a subsample
  3. Combine treatments

# Non-Interaction Effects of Interest

1. Average Treatment Combination Effect (ATCE):
    - Average effect of treatment combination $(A, B) = (a_\ell, b_m)$ relative to the baseline condition $(A, B) = (a_0, b_0)$

$$\tau(a_\ell, b_m; a_0, b_0) \equiv \mathbb{E}\{Y(a_\ell, b_m) - Y(a_0, b_0)\}$$

    - Which treatment combination is most efficacious?

2. Average Marginal Treatment Effect (AMTE; Hainmueller et al. 2014):
    - Average effect of treatment $A = a_\ell$ relative to the baseline condition $A = a_0$ averaging over the other treatment $B$

$$\psi(a_\ell, a_0) \equiv \int_{\mathcal{B}} \mathbb{E}\{Y(a_\ell, B) - Y(a_0, B)\} dF(B)$$

    - Which treatment is effective on average?

# The Conventional Approach to Causal Interaction

- Average Treatment Interaction Effect (ATIE):

$$\xi(a_\ell, b_m; a_0, b_0) \equiv \mathbb{E}\{Y(a_\ell, b_m) - Y(a_0, b_m) - Y(a_\ell, b_0) + Y(a_0, b_0)\}$$

- Conditional effect interpretation:

$$\underbrace{\mathbb{E}\{Y(a_\ell, b_m) - Y(a_0, b_m)\}}_{\text{Effect of } A = a_\ell \text{ when } B = b_m} - \underbrace{\mathbb{E}\{Y(a_\ell, b_0) - Y(a_0, b_0)\}}_{\text{Effect of } A = a_\ell \text{ when } B = b_0}$$

- Interactive effect interpretation:

$$\underbrace{\tau(a_\ell, b_m; a_0, b_0)}_{\text{ATCE}} - \underbrace{\mathbb{E}\{Y(a_\ell, b_0) - Y(a_0, b_0)\}}_{\text{Effect of } A = a_\ell \text{ when } B = b_0} - \underbrace{\mathbb{E}\{Y(a_0, b_m) - Y(a_0, b_0)\}}_{\text{Effect of } B = b_m \text{ when } A = a_0}$$

- Estimation: Linear regression with interaction terms

# The New Causal Interaction Effect

- Average Marginal Treatment Interaction Effect (**AMTIE**):

$$\pi(a_\ell, b_m; a_0, b_0) \equiv \underbrace{\tau(a_\ell, b_m; a_0, b_0)}_{\text{ATCE of } (A,B)=(a_\ell, b_m)} - \underbrace{\psi(a_\ell, a_0)}_{\text{AMTE of } a_\ell} - \underbrace{\psi(b_m, b_0)}_{\text{AMTE of } b_m}$$

- Interactive effect interpretation: additional effect induced by $A = a_\ell$ and $B = b_m$ together beyond the separate effect of $A = a_\ell$ and that of $B = b_m$
- Compare this with ATIE:

$$\underbrace{\tau(a_\ell, b_m; a_0, b_0)}_{\text{ATCE}} - \underbrace{\mathbb{E}\{Y(a_\ell, b_0) - Y(a_0, b_0)\}}_{\text{Effect of } A=a_\ell \text{ when } B=b_0} - \underbrace{\mathbb{E}\{Y(a_0, b_m) - Y(a_0, b_0)\}}_{\text{Effect of } B=b_m \text{ when } A=a_0}$$

- We prove that the **AMTIE**s are both *interval and order invariant*
- The **AMTIE**s do depend on the distribution of treatment assignment
  1. specified by one's experimental design
  2. motivated by the target population

# AMTIE is Invariant to the Choice of Baseline Condition

| | Education | | | | | | |
|---|---|---|---|---|---|---|---|
| **Job experience** | None | 4th grade | 8th grade | High school | Two-year college | College | Graduate |
| None | 0 | −0.004 | −0.028 | −0.035 | −0.031 | 0.012 | −0.010 |
| 1–2 years | −0.001 | −0.001 | −0.025 | −0.040 | 0.024 | −0.009 | −0.044 |
| 3–5 years | −0.040 | −0.019 | −0.042 | 0.031 | −0.026 | −0.022 | 0.024 |
| $>$ 5 years | −0.014 | −0.031 | 0.041 | −0.011 | −0.021 | −0.036 | −0.024 |

# AMTIE is Invariant to the Choice of Baseline Condition

| | Education | | | | | | |
|---|---|---|---|---|---|---|---|
| **Job experience** | None | 4th grade | 8th grade | High school | Two-year college | College | Graduate |
| None | 0.024 | 0.020 | −0.004 | −0.011 | −0.007 | 0.036 | 0.014 |
| 1–2 years | 0.023 | 0.023 | −0.001 | −0.016 | 0.048 | 0.015 | −0.020 |
| 3–5 years | −0.016 | 0.005 | −0.018 | 0.055 | −0.002 | 0.002 | 0.048 |
| $>$ 5 years | 0.010 | −0.007 | 0.065 | 0.013 | 0.003 | −0.012 | 0 |

1. The **AMTIE** is a linear function of the ATIEs:

$$\pi(a_\ell, b_m; a_0, b_0) = \xi(a_\ell, b_m; a_0, b_0) - \sum_{a \in \mathcal{A}} \Pr(A_i = a)\, \xi(a, b_m; a_0, b_0) - \sum_{b \in \mathcal{B}} \Pr(B_i = b)\, \xi(a_\ell, b; a_0, b_0)$$

2. The ATIE is also a linear function of the **AMTIE**s:

$$\xi(a_\ell, b_m; a_0, b_0) = \pi(a_\ell, b_m; a_0, b_0) - \pi(a_\ell, b_0; a_0, b_0) - \pi(a_0, b_m; a_0, b_0)$$

- Absence of causal interaction:
  All of the **AMTIE**s are zero if and only if all of the ATIEs are zero
- The **AMTIE**s can be estimated by first estimating the ATIEs

# Higher-order Causal Interaction

- $J$ factorial treatments: $\mathbf{T} = (T_1, \ldots, T_J)$
- Assumptions:
  1. Full factorial design

$$Y(\mathbf{t}) \quad \perp\!\!\!\perp \quad \mathbf{T} \quad \text{and} \quad \Pr(\mathbf{T} = \mathbf{t}) \ > \ 0 \quad \text{for all } \mathbf{t}$$

  2. Independent treatment assignment

$$T_j \quad \perp\!\!\!\perp \quad \mathbf{T}_{-j} \quad \text{for all } j$$

- Assumption 2 is not necessary for identification but considerably simplifies estimation

- We are interested in the $K$-way interaction where $K \leq J$
- We extend all the results for the 2-way interaction to this general case

# Difficulty of Interpreting the Higher-order ATIE

- Generalize the 2-way ATIE by marginalizing the other treatments $\underline{\mathbf{T}}^{1:2}$

$$\xi_{1:2}(t_1, t_2; t_{01}, t_{02}) \equiv \int \mathbb{E}\left\{ Y(t_1, t_2, \underline{\mathbf{T}}^{1:2}) - Y(t_{01}, t_2, \underline{\mathbf{T}}^{1:2}) - Y(t_1, t_{02}, \underline{\mathbf{T}}^{1:2}) + Y(t_{01}, t_{02}, \underline{\mathbf{T}}^{1:2}) \right\} dF(\underline{\mathbf{T}}^{1:2})$$

- In the literature, the 3-way ATIE is defined as

$$\xi_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03}) \equiv \underbrace{\xi_{1:2}(t_1, t_2; t_{01}, t_{02} \mid T_3 = t_3)}_{\text{2-way ATIE when } T_3 = t_3} - \underbrace{\xi_{1:2}(t_1, t_2; t_{01}, t_{02} \mid T_3 = t_{03})}_{\text{2-way ATIE when } T_3 = t_{03}}$$

- Higher-order ATIEs are similarly defined sequentially
- This representation is based on the conditional effect interpretation
- Problem: the conditional effect of conditional effects!

# The $K$-way Average Marginal Treatment Interaction Effect

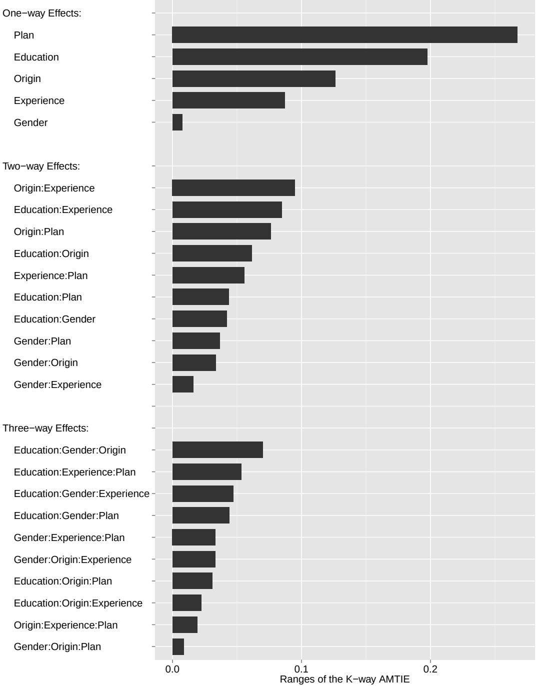
- Definition: the difference between the ATCE and the sum of lower-order **AMTIE**s
- Interactive effect interpretation
- Example: 3-way **AMTIE**, $\pi_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03})$, equals

$$\underbrace{\tau_{1:3}(t_1, t_2, t_3; t_{01}, t_{02}, t_{03})}_{\text{ATCE}}$$
$$- \underbrace{\left\{\pi_{1:2}(t_1, t_2; t_{01}, t_{02}) + \pi_{2:3}(t_2, t_3; t_{02}, t_{03}) + \pi_{1:3}(t_1, t_3; t_{01}, t_{03})\right\}}_{\text{sum of 2-way AMTIEs}}$$
$$- \underbrace{\left\{\psi(t_1; t_{01}) + \psi(t_2; t_{02}) + \psi(t_3; t_{03})\right\}}_{\text{sum of (1-way) AMTEs}}$$

- Properties:
  1. $K$-way ATCE $=$ the sum of all $K$-way and lower-order **AMTIE**s
  2. Interval and order invariance to the baseline condition
  3. Derive the relationships between the **AMTIE**s and ATIEs for any order

# Empirical Analysis of the Immigration Survey Experiment

- 5 factors (`gender`$^2$, `education`$^7$, `origin`$^{10}$, `experience`$^4$, `plan`$^4$)
  1. full factorial design assumption
  2. computational tractability
- Matched-pair conjoint analysis: randomly choose one profile
- Binary outcome: whether a profile is selected
- Model with one-way, two-way, and three-way interaction terms
- $p = 1,575$ and $n = 6,980$
- Curse of dimensionality $\Longrightarrow$ sparcity assumption
- Support vector machine with a lasso constraint (Imai & Ratkovic, 2013)
- Under-identified model that includes baseline conditions
- 99 non-zero and $1,476$ zero coefficients
- Cross-validation for selecting a tuning parameter
- `FindIt: Finding heterogeneous treatment effects`




- Range of **AMTIE**s: importance of each factor and factor interaction
- Sparcity-of-effects principle
- `gender` appears to play a significant role in three-way interactions

# Decomposing the Average Treatment Combination Effect

- Two-way effect example (`origin` $\times$ `experience`):

$$\underbrace{\tau(\texttt{Somalia, 1-2 years; India, None})}_{-3.74} \qquad (n = 168; \quad n = 155)$$

$$= \underbrace{\psi(\texttt{Somalia}; \texttt{India})}_{-5.14} + \underbrace{\psi(\texttt{1} - \texttt{2years}; \texttt{None})}_{5.12} + \underbrace{\pi(\texttt{Somalia}, \texttt{1} - \texttt{2years}; \texttt{India}, \texttt{None})}_{-3.72}$$

- Three-way examples (`education` $\times$ `gender` $\times$ `origin`):

$$\underbrace{\tau(\texttt{Graduate}, \texttt{Male}, \texttt{India}; \texttt{Graduate}, \texttt{Female}, \texttt{India})}_{7.46} \qquad (n = 52; \quad n = 40)$$

$$= \underbrace{\psi(\texttt{Male}; \texttt{Female})}_{-0.77} + \underbrace{\pi(\texttt{Graduate}, \texttt{Male}; \texttt{Graduate}, \texttt{Female})}_{-0.34}$$

$$+ \underbrace{\pi(\texttt{Male}, \texttt{India}; \texttt{Female}, \texttt{India})}_{1.56} + \underbrace{\pi(\texttt{Graduate}, \texttt{Male}, \texttt{India}; \texttt{Graduate}, \texttt{Female}, \texttt{India})}_{7.01}$$

# Concluding Remarks

- Interaction effects play an essential role in causal heterogeneity
  1. moderation
  2. causal interaction
- Two interpretations of causal interaction
  1. conditional effect interpretation (problematic in high dimension)
  2. interactive effect interpretation
- Average Marginal Treatment Interaction Effect
  1. interactive effect in high-dimension
  2. invariant to baseline condition
  3. enables effect decomposition
  4. $\rightsquigarrow$ effective analysis of interactions in high-dimension
- Estimation challenges in high dimension
  1. group lasso, hierarchical interaction
  2. post-selection inference

# References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