

A New Automated Redistricting Simulator Using Markov Chain Monte Carlo

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Joint work with Benjamin Fifield, Michael Higgins, and Alexander Tarr

Motivation

- Redistricting as a central element of representative democracy
- Redistricting may affect:
 - Representation (Gelman and King 1994, McCarty *et. al* 2009)
 - Turnout (Gay 2001, Baretto 2004)
 - Incumbency advantage (Abramowitz *et. al* 2006)
- Substantive researchers simulate redistricting plans to:
 - detect gerrymandering
 - assess impact of constraints (e.g., population, compactness, race)
- Many optimization methods but surprisingly few simulation methods
- Standard algorithm has no theoretical justification
- Need a simulation method that:
 - ① samples uniformly from a target population of redistricting maps
 - ② incorporates common constraints
 - ③ scales to redistricting problems of moderate and large size

Overview of the Talk

- ➊ Explain the difficulties of simulating redistricting plans
- ➋ Propose new **Markov chain Monte Carlo** algorithms
- ➌ Validate the algorithms on a small-scale data example
- ➍ Present empirical analyses for New Hampshire and Mississippi

Characterizing the Distribution of Valid Redistricting Plans

- Scholars want to characterize the *distribution* of redistricting plans under various constraints
- Valid redistricting plans must have:
 - geographically **contiguous** districts
 - districts with **equal population**
- Other constraints of interest: compactness, community boundary, etc.
- Naive Approach 1: Enumeration
 - Can't enumerate all plans (too many)
 - Enumerating only valid plans is not trivial
- Naive Approach 2: Random assignment
 - Too few plans will have equal population
 - Too few plans will be contiguous

The Standard Simulation Algorithm

- **Random seed-and-grow** algorithm (Cirincione *et. al* 2000, Altman & McDonald 2011, Chen & Rodden 2013):
 - ➊ Randomly choose a precinct as a “seed” for each district
 - ➋ Identify precincts adjacent to each seed
 - ➌ Randomly select adjacent precinct to merge with the seed
 - ➍ Repeat steps 2 & 3 until all precincts are assigned
 - ➎ Swap precincts around borders to achieve population parity
- Modify Step 3 to incorporate compactness
- No theoretical properties known
- The resulting sample may not be representative of the population
- “Local” exploration is difficult

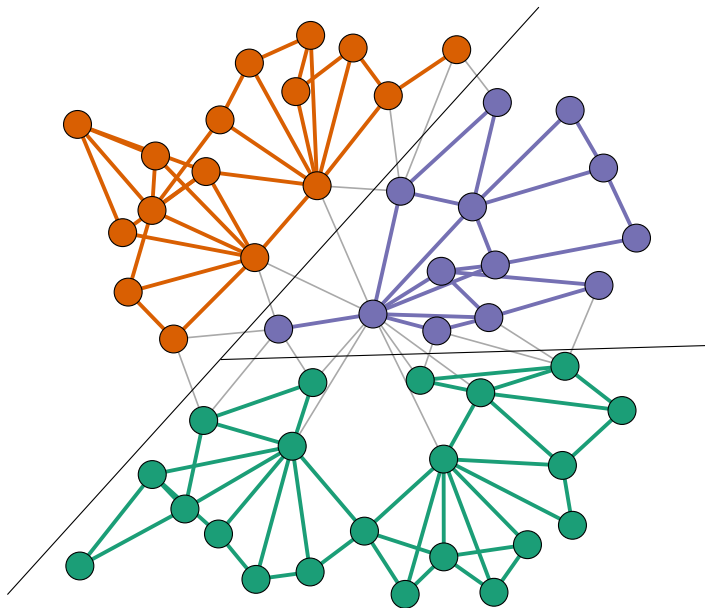
The Proposed Automated Redistricting Simulator

- Independent sampling is difficult
- Can sample uniformly from the target distribution
- Markov chain Monte Carlo algorithm
- Start with a valid plan and then swap precincts in a certain way

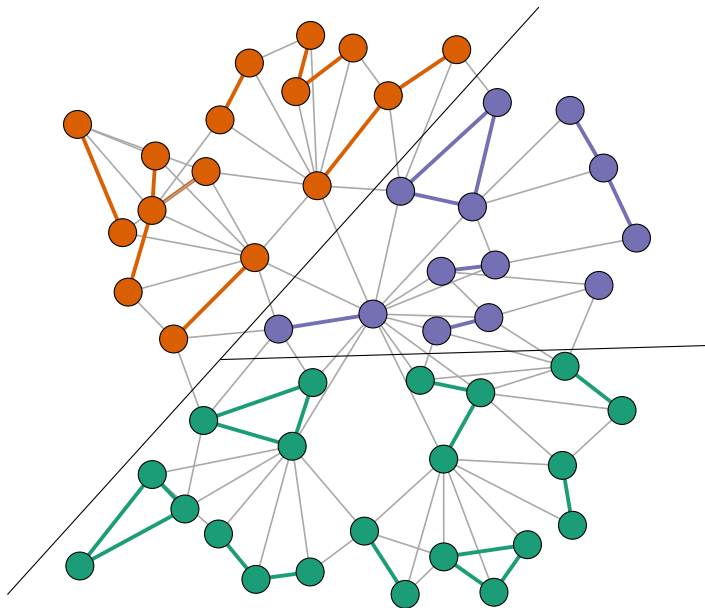
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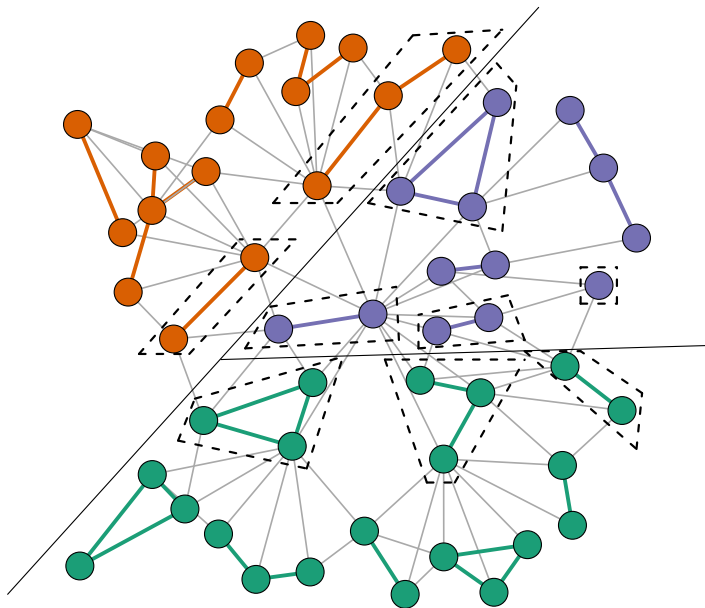
Redistricting as a **Graph-Cut** Problem



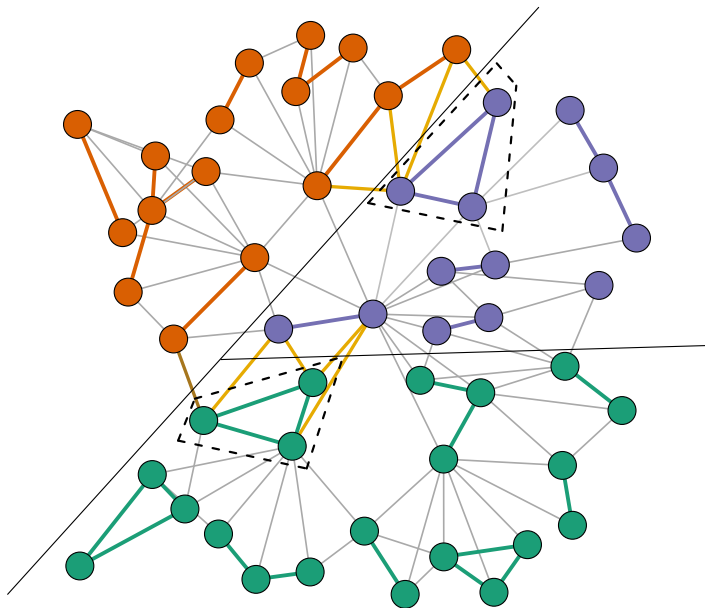
Step 1: Independently “Turn On” Each Edge with Prob. q



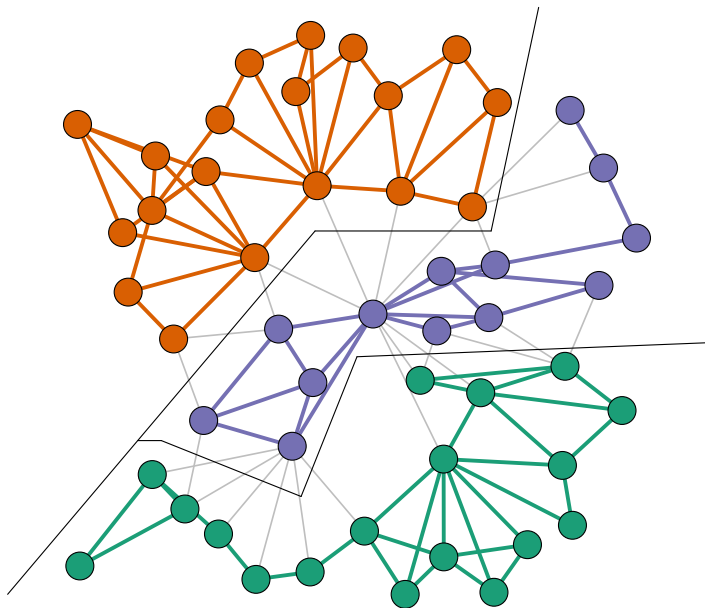
Step 2: Gather Connected Components on Boundaries



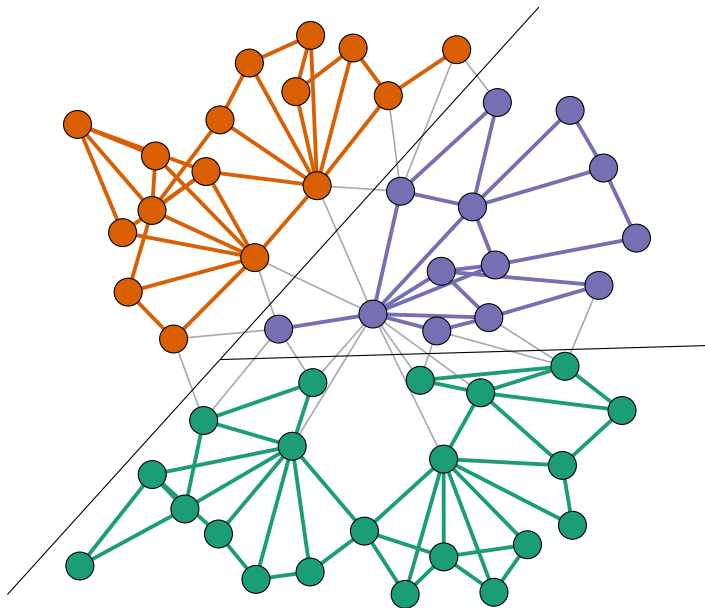
Step 3: Select Subsets of Components and Propose Swaps



Step 4: Accept or Reject the Proposal



Step 4: Accept or Reject the Proposal



The Theoretical Property of the Algorithm

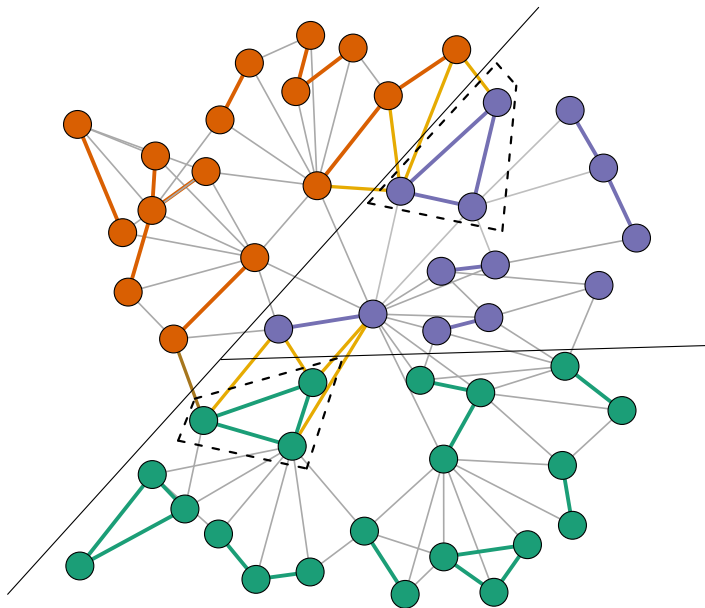
- We prove that the algorithm samples *uniformly* from the population of all valid redistricting plans
- An extension of the **Swendsen-Wang** algorithm (Barbu & Zhu, 2005)
- **Metropolis-Hastings** move from plan $\mathbf{v}_{t-1} \rightarrow \mathbf{v}_t^*$:

$$\begin{aligned}\alpha(\mathbf{v}_{t-1} \rightarrow \mathbf{v}_t^*) &= \min \left(1, \frac{\pi(\mathbf{v}_t^* \rightarrow \mathbf{v}_{t-1})}{\pi(\mathbf{v}_{t-1} \rightarrow \mathbf{v}_t^*)} \right) \\ &\approx \min \left(1, (1-q)^{|B(C^*, \mathbf{v})| - |B(C^*, \mathbf{v}^*)|} \right)\end{aligned}$$

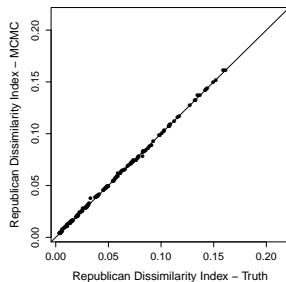
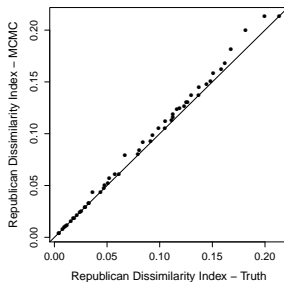
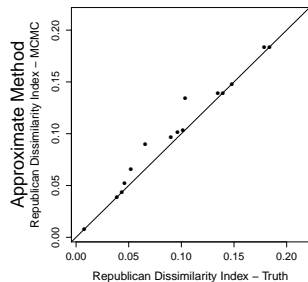
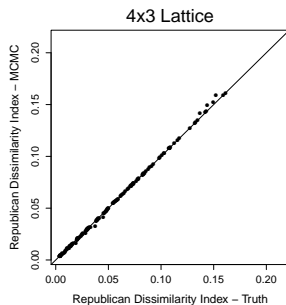
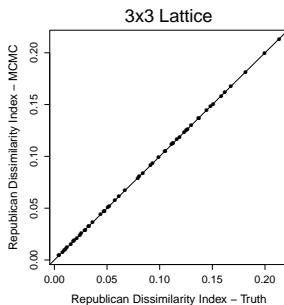
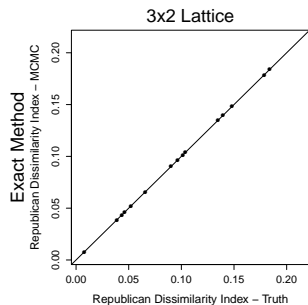
where q is the edge cut probability and $|B(C^*, \mathbf{v})|$ is # of edges between connected component and its assigned district in redistricting plan $\mathbf{v} \rightsquigarrow$ **Easy to calculate**

- Exact Metropolis ratio is too costly to evaluate

The Theoretical Property of the Algorithm



Accuracy of Approximation with a Long Chain



Incorporating a Population Constraint

- Want to sample plans where

$$\psi(V_k) = \left| \frac{p_k}{\bar{p}} - 1 \right| \leq \epsilon$$

where p_k is population of district k , \bar{p} is average district population, ϵ is strength of constraint

- Strategy 1:** Only propose “valid” swaps \rightsquigarrow slow mixing
- Strategy 2:** Oversample certain plans and then reweight
 - Sample from target distribution f rather than the uniform distribution:

$$f(\mathbf{v}) \propto g(\mathbf{v}) = \exp \left(-\beta \sum_{V_k \in \mathbf{v}} \psi(V_k) \right)$$

where $\beta \geq 0$ and $\psi(V_k)$ is deviation from parity for district V_k

- (Approximate) Acceptance probability is still easy to calculate,

$$\alpha(\mathbf{v} \rightarrow \mathbf{v}^*) \approx \min \left(1, \frac{g(\mathbf{v}^*)}{g(\mathbf{v})} \cdot (1 - q)^{|B(C^*, \mathbf{v})| - |B(C^*, \mathbf{v}^*)|} \right)$$

- Discard invalid plans and reweight the rest by $1/g(\mathbf{v})$

Additional Constraints

- 1 **Compactness** (Fryer and Holden 2011):

$$\psi(V_k) \propto \sum_{i,j \in V_k, i < j} p_i p_j d_{ij}^2$$

where d_{ij} is the distance between precincts i, j

- 2 **Similarity to the adapted plan:**

$$\psi(V_k) = \left| \frac{r_k}{r_k^*} - 1 \right|$$

where r_k (r_k^*) is the # of precincts in V_k (V_k of the adapted plan)

- Any criteria where constraint can be evaluated at each district

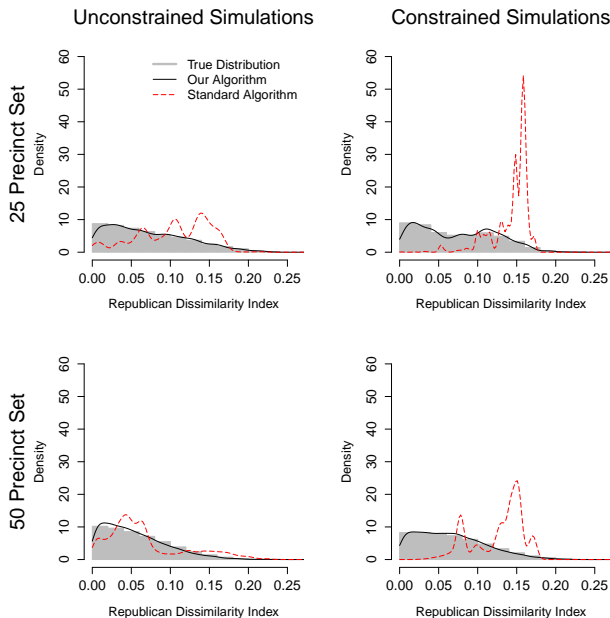
Improving the Mixing of the Algorithm

- Single iteration of the proposed algorithm runs very quickly
 - But, like any MCMC algorithm, convergence may take a long time
- ❶ Swapping multiple connected components
 - more effective than increasing q
 - but still leads to low acceptance ratio
 - ❷ **Simulated tempering** (Geyer and Thompson, 1995)
 - Lower and raise the “temperature” parameter β as part of MCMC
 - Explores low temperature space before visiting high temperature space
 - ❸ **Parallel tempering** (Geyer 1991)
 - Run multiple chains of the algorithm with different temperatures
 - Use the Metropolis criterion to swap temperatures with adjacent chains

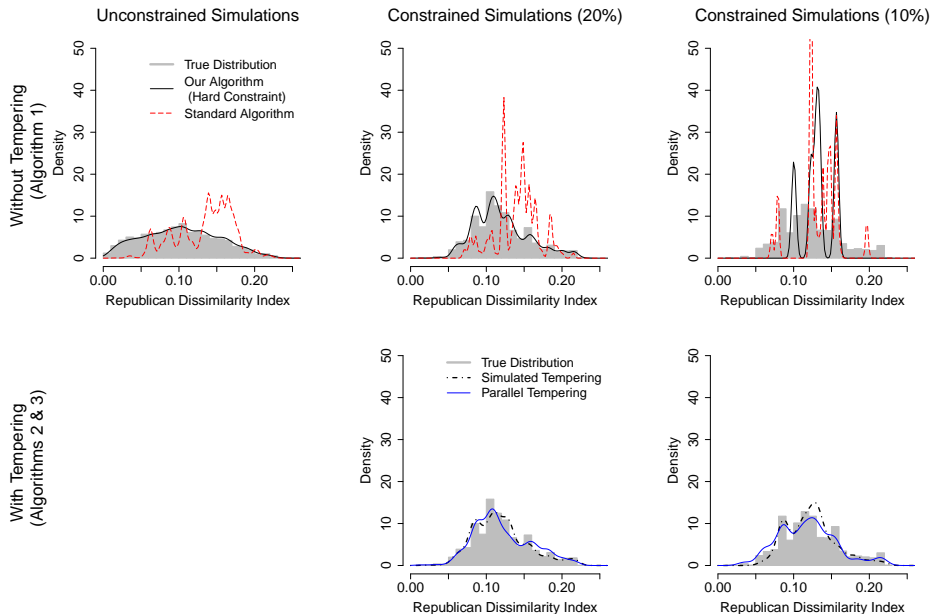
A Small-Scale Validation Study

- Evaluate algorithms when all valid plans can be enumerated
- # of precincts: 25 and 50
- # of districts: 2 and 3 for the 25 set, and 2 for the 50 set
- With and without a “hard” population constraint of 20% within parity
- Also, consider simulated and parallel tempering
- Comparison with the “random seed-and-grow” algorithm via the “industry-standard” BARD package (Altman & McDonald 2011)
- 10,000 draws for each algorithm

Our Algorithm vs. Standard Algorithm

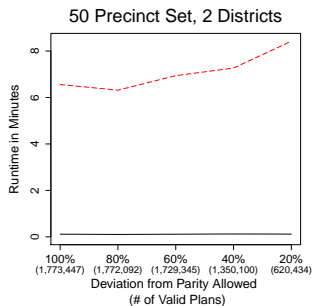
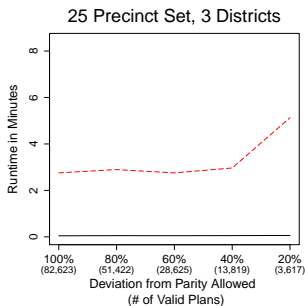
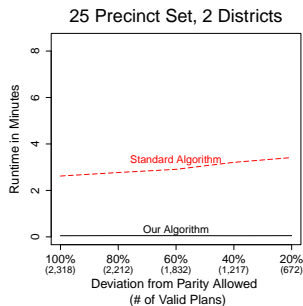


Simulated and Parallel Tempering



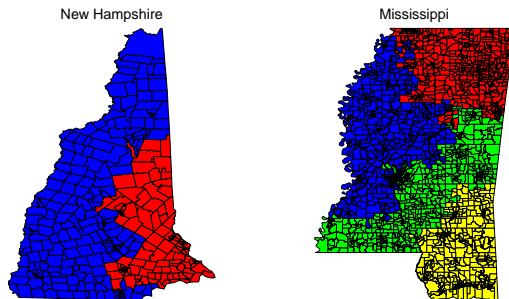
Runtime Comparison

- Run each algorithm for 10,000 simulations under different population constraints

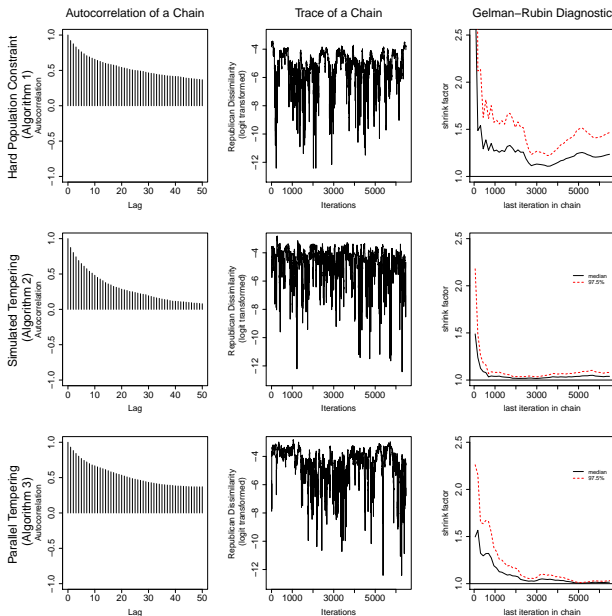


An Empirical Study

- Apply algorithm to state election data:
 - ① New Hampshire: 2 congressional districts, 327 precincts
 - ② Mississippi: 4 congressional districts, 1,969 precincts
 - ③ 1% (NH) and 5% (MS) deviation from population parity
- Convergence diagnostics:
 - ① Autocorrelation
 - ② Trace plot
 - ③ Gelman-Rubin multiple chain diagnostic

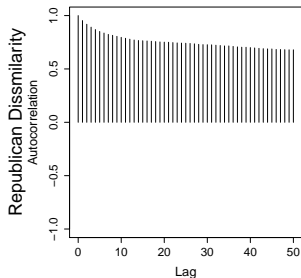


New Hampshire: Tempering Works Better

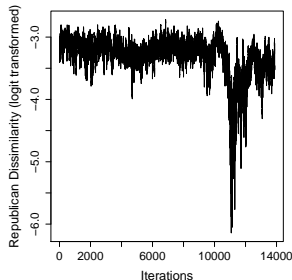


Mississippi: Parallel Tempering, More Challenging Case

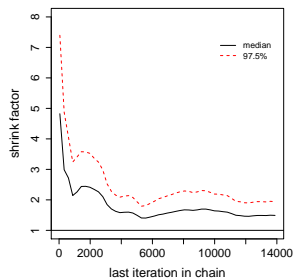
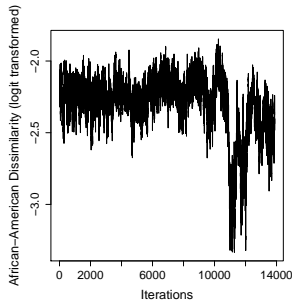
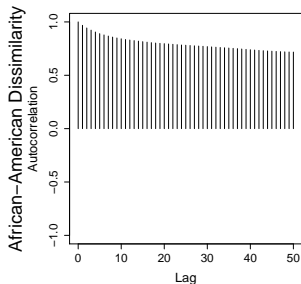
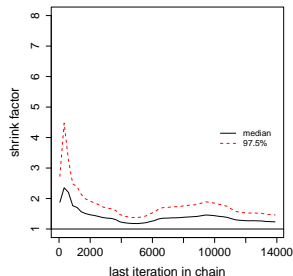
Autocorrelation of a Chain



Trace of a Chain



Gelman–Rubin Diagnostic

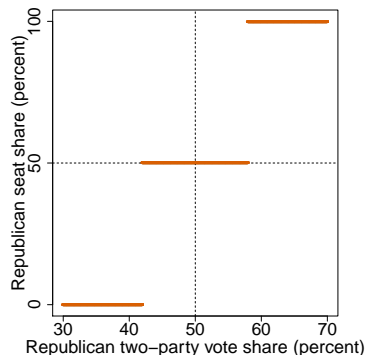
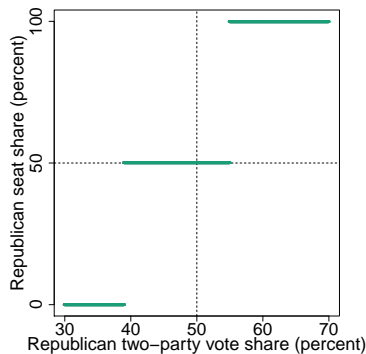


Redistricting Plans that are Similar to the Adapted Plan

- Question: How does the partisan bias of the adapted plan compare with that of similar plans?
- Two measures:
 - ① Number of Republican winners under each plan
 - ② Partisan bias (Gelman & King, 1994): Deviation from partisan symmetry under each plan
- Geweke diagnostics: comparison of means within a single chain

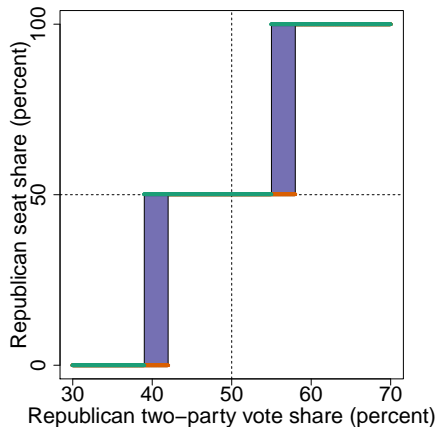
Evaluating Partisan Bias

- Empirical and Symmetric Seats-Votes Curves

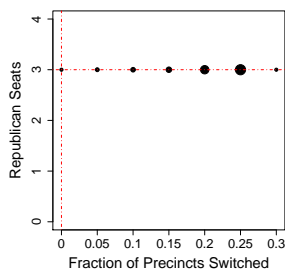
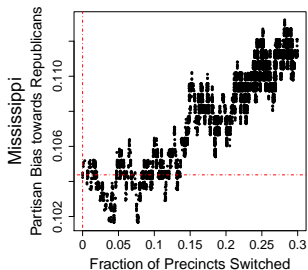
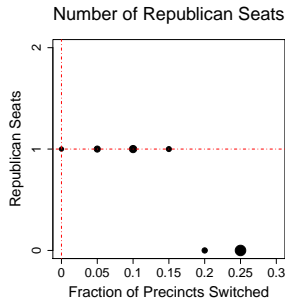
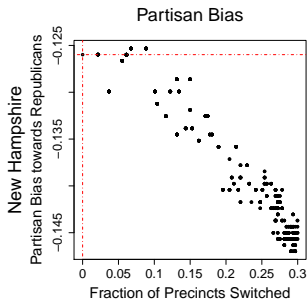


Evaluating Partisan Bias

- Absolute Deviation from Partisan Symmetry



Partisan Implications of “Local Exploration”



Concluding Remarks

- Scholars use simulations to characterize the distribution of redistricting plans
- Many optimization algorithms but very few simulation methods
- Commonly used algorithms are adhoc
- We propose a new MCMC algorithm that has:
 - better theoretical properties
 - superior speed
 - better performance in validation and empirical studies
- Future research:
 - Apply the method to historical redistricting data
 - Explore alternative ways to tackle large scale redistricting problems

References

- ❶ Paper at <http://imai.princeton.edu/research/redist.html>
- ❷ R package at <https://github.com/redistricting/redist>
- ❸ Comments and suggestions: send them to kimai@princeton.edu