Understanding and Improving Linear Fixed Effects Regression Models for Causal Inference

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Motivation

- Fixed effects models are a primary workhorse for causal inference
- Researchers use them for stratified randomized experiments
- Also used to adjust for unobservables in observational studies:
  - "Good instruments are hard to find ..., so we'd like to have other tools to deal with unobserved confounders. This chapter considers ... strategies that use data with a time or cohort dimension to control for unobserved but fixed omitted variables" (Angrist & Pischke, *Mostly Harmless Econometrics*)
  - "fixed effects regression can scarcely be faulted for being the bearer of bad tidings" (Green *et al.*, *Dirty Pool*)
- Fixed effects models are often said to be superior to matching estimators because the latter can only adjust for observables
- **Question:** What are the exact causal assumptions underlying fixed effects regression models?

Main Results

1. Standard (one-way and two-way) FE estimators are equivalent to particular matching estimators
2. Common belief that FE models adjust for unobservables but matching does not is wrong
3. Identify the information used implicitly to estimate counterfactual outcomes under FE models
4. Identify potential sources of bias and inefficiency in FE estimators
5. Propose simple ways to improve FE estimators using weighted FE regression
6. Within-unit matching, first differencing, propensity score weighting, difference-in-differences are all equivalent to weighted FE model with different regression weights
7. Offer a specification test for the standard FE model

Matching and Regression in Cross-Section Settings

Units	**1**	**2**	**3**	**4**	**5**
Treatment status	**T**	**T**	**C**	**C**	**T**
Outcome	Y_1	Y_2	Y_3	Y_4	Y_5

- Estimating the Average Treatment Effect (ATE) via matching:

$$\begin{array}{rcl}
Y_1 & - & \frac{1}{2}(Y_3 + Y_4) \\
Y_2 & - & \frac{1}{2}(Y_3 + Y_4) \\
\frac{1}{3}(Y_1 + Y_2 + Y_5) & - & Y_3 \\
\frac{1}{3}(Y_1 + Y_2 + Y_5) & - & Y_4 \\
Y_5 & - & \frac{1}{2}(Y_3 + Y_4)
\end{array}$$

Matching Representation of Simple Regression

- Cross-section simple linear regression model:

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

- Binary treatment: $X_i \in \{0, 1\}$
- Equivalent matching estimator:

$$\hat{\beta} = \frac{1}{N} \sum_{i=1}^{N} \left(\widehat{Y_i(1)} - \widehat{Y_i(0)} \right)$$

where

$$\widehat{Y_i(1)} = \begin{cases} Y_i & \text{if } X_i = 1 \\ \frac{1}{\sum_{i'=1}^{N} X_{i'}} \sum_{i'=1}^{N} X_{i'} Y_{i'} & \text{if } X_i = 0 \end{cases}$$

$$\widehat{Y_i(0)} = \begin{cases} \frac{1}{\sum_{i'=1}^{N} (1 - X_{i'})} \sum_{i'=1}^{N} (1 - X_{i'}) Y_{i'} & \text{if } X_i = 1 \\ Y_i & \text{if } X_i = 0 \end{cases}$$

- Treated units matched with the average of non-treated units

One-Way Fixed Effects Regression

- Simple (one-way) FE model:

$$Y_{it} = \alpha_i + \beta X_{it} + \epsilon_{it}$$

- Commonly used by applied researchers:
  - Stratified randomized experiments (Duflo *et al.* 2007)
  - Stratification/matching in observational studies
  - Panel data, both experimental and observational

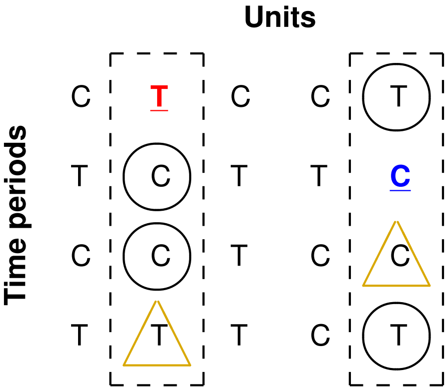
- $\hat{\beta}_{FE}$ may be biased for the ATE even if X_{it} is exogenous within each unit: converges to the weighted average of conditional ATEs:

$$\hat{\beta}_{FE} \xrightarrow{p} \frac{\mathbb{E}\{\Delta(X_i)\, \sigma_i^2\}}{\mathbb{E}(\sigma_i^2)}$$

  where $\sigma_i^2 = \sum_{t=1}^{T} (X_{it} - \overline{X}_i)^2 / T$

- How are counterfactual outcomes estimated under the FE model?
- Unit fixed effects $\Longrightarrow$ within-unit comparison

Mismatches in One-Way Fixed Effects Model



- T: treated observations
- C: control observations
- **Circles**: Proper matches
- **Triangles**: "Mismatches" $\Longrightarrow$ attenuation bias

Matching Representation of Fixed Effects Regression

Proposition 1

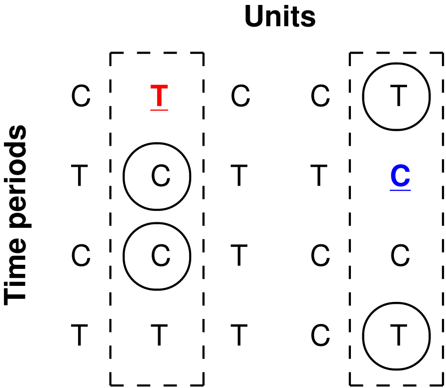
$$\hat{\beta}^{FE} = \frac{1}{K}\left\{\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)}\right)\right\},$$

$$\widehat{Y_{it}(x)} = \begin{cases} Y_{it} & \text{if } X_{it} = x \\ \frac{1}{T-1}\sum_{t' \neq t} Y_{it'} & \text{if } X_{it} = 1 - x \end{cases} \quad \textit{for} \quad x = 0, 1$$

$$K = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\{X_{it} \cdot \frac{1}{T-1}\sum_{t' \neq t}(1 - X_{it'}) + (1 - X_{it}) \cdot \frac{1}{T-1}\sum_{t' \neq t} X_{it'}\right\}.$$

- K: average proportion of proper matches across all observations
- More mismatches $\Longrightarrow$ larger adjustment
- Adjustment is required except very special cases
- “Fixes” attenuation bias but this adjustment is not sufficient
- Fixed effects estimator is a special case of matching estimators

Unadjusted Matching Estimator



- Consistent if the treatment is exogenous within each unit
- Only equal to fixed effects estimator if heterogeneity in either treatment assignment or treatment effect is non-existent

Unadjusted Matching as **Weighted** FE Estimator

Proposition 2

The unadjusted matching estimator

$$\hat{\beta}^M = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)}\right)$$

where

$$\widehat{Y_{it}(1)} = \begin{cases} Y_{it} & \text{if } X_{it} = 1 \\ \frac{\sum_{t'=1}^{T} X_{it'} Y_{it'}}{\sum_{t'=1}^{T} X_{it'}} & \text{if } X_{it} = 0 \end{cases} \quad \text{and} \quad \widehat{Y_{it}(0)} = \begin{cases} \frac{\sum_{t'=1}^{T} (1 - X_{it'}) Y_{it'}}{\sum_{t'=1}^{T} (1 - X_{it'})} & \text{if } X_{it} = 1 \\ Y_{it} & \text{if } X_{it} = 0 \end{cases}$$

is equivalent to the weighted fixed effects model

$$(\hat{\alpha}^M, \hat{\beta}^M) = \underset{(\alpha,\beta)}{\arg\min} \sum_{i=1}^{N}\sum_{t=1}^{T} W_{it}(Y_{it} - \alpha_i - \beta X_{it})^2$$

$$W_{it} \equiv \begin{cases} \frac{T}{\sum_{t'=1}^{T} X_{it'}} & \text{if} \quad X_{it} = 1, \\ \frac{T}{\sum_{t'=1}^{T} (1 - X_{it'})} & \text{if} \quad X_{it} = 0. \end{cases}$$

Equal Weights

Treatment

C	**T**	C	C	T
T	C	T	T	**C**
C	C	T	C	C
T	T	T	C	T

Weights

0	**1**	0	0	$\frac{1}{2}$
0	$\frac{1}{2}$	0	0	**1**
0	$\frac{1}{2}$	0	0	0
0	0	0	0	$\frac{1}{2}$

Different Weights

Treatment

C	T	C	C	T
T	C	T	T	C
C	C	T	C	C
T	T	T	C	T

Weights

0	1	0	0	$\frac{3}{4}$
0	$\frac{2}{3}$	0	0	1
0	$\frac{1}{3}$	0	0	0
0	0	0	0	$\frac{1}{4}$

- Any within-unit matching estimator leads to weighted fixed effects regression with particular weights
- We derive regression weights given a matching estimator for various quantities (ATE, ATT, etc.)

Theorem: General Equivalence between Weighted Fixed Effects and Matching Estimators

General matching estimator

$$\tilde{\beta}^M = \frac{1}{\sum_{i=1}^N \sum_{t=1}^T C_{it}} \sum_{i=1}^N \sum_{t=1}^T C_{it} \left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)} \right)$$

where $0 \leq C_{it} < \infty$, $\sum_{t=1}^T \sum_{i=1}^N C_{it} > 0$,

$$\widehat{Y_{it}(1)} = \begin{cases} Y_{it} & \text{if } X_{it} = 1 \\ \sum_{t'=1}^T v_{it}^{it'} X_{it'} Y_{it'} & \text{if } X_{it} = 0 \end{cases}$$

$$\widehat{Y_{it}(0)} = \begin{cases} \sum_{t'=1}^T v_{it}^{it'} (1 - X_{it'}) Y_{it'} & \text{if } X_{it} = 1 \\ Y_{it} & \text{if } X_{it} = 0 \end{cases}$$

$$\sum_{t'=1}^T v_{it}^{it'} X_{it'} = \sum_{t'=1}^T v_{it}^{it'} (1 - X_{it'}) = 1$$

is equivalent to the weighted one-way fixed effects estimator

$$W_{it} = \sum_{i'=1}^N \sum_{t'=1}^T w_{it}^{i't'} \quad \text{and} \quad w_{it}^{i't'} = \begin{cases} C_{it} & \text{if } (i,t) = (i',t') \\ v_{it}^{it'} C_{i't'} & \text{if } (i,t) \in \mathcal{M}_{i't'} \\ 0 & \text{otherwise.} \end{cases}$$

First Difference Estimator

- $\Delta Y_{it} = \beta \Delta X_{it} + \epsilon_{it}$ where $\Delta Y_{it} = Y_{it} - Y_{i,t-1}$, $\Delta X_{it} = X_{it} - X_{i,t-1}$

Treatment

C	**T**	C	C	T
T	C	T	T	**C**
C	C	T	C	C
T	T	T	C	T

Weights

0	**1**	0	0	0
0	1	0	0	**0**
0	0	0	0	0
0	0	0	0	0

- First-difference = matching = weighted one-way fixed effects

Adjusting for Observation-Specific Confounders Z_{it}

1. Regression-adjusted matching:

$$Y_{it} - \widehat{g(Z_{it})} \quad \text{where} \quad g(z) = \mathbb{E}(Y_{it} \mid X_{it} = 0, Z_{it} = z)$$

2. Direct regression adjustment:

$$\underset{(\alpha,\beta,\delta)}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} W_{it}(Y_{it} - \alpha_i - \beta X_{it} - \delta^\top Z_{it})^2$$

- *Ex post* interpretation: $Y_{it} - \hat{\delta}^\top Z_{it} \; = \; \alpha_i + \beta X_{it} + \epsilon_{it}$

3. Inverse-propensity score weighting with normalized weights

$$\hat{\beta}^W \quad = \quad \frac{1}{N} \sum_{i=1}^{N} \left\{ \sum_{t=1}^{T} \frac{X_{it} Y_{it}}{\hat{\pi}(Z_{it})} \Big/ \sum_{t=1}^{T} \frac{X_{it}}{\hat{\pi}(Z_{it})} - \sum_{t=1}^{T} \frac{(1 - X_{it}) Y_{it}}{1 - \hat{\pi}(Z_{it})} \Big/ \sum_{t=1}^{T} \frac{(1 - X_{it})}{1 - \hat{\pi}(Z_{it})} \right\}$$

where $\pi(Z_{it}) = \Pr(X_{it} = 1 \mid Z_{it})$ is the propensity score

- within-unit weighting followed by across-units averaging

Propensity Score Weighting Estimator is Equivalent to Transformed Weighted FE Estimator

Proposition 3

$$(\hat{\alpha}^W, \hat{\beta}^W) \;=\; \underset{(\alpha,\beta)}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} W_{it}(Y_{it}^* - \alpha_i - \beta X_{it})^2$$

where the transformed outcome Y_{it}^* is,

$$Y_{it}^* \;=\; \begin{cases} \frac{\left(\sum_{t'=1}^{T} X_{it'}\right) Y_{it}}{\hat{\pi}(Z_{it})} \Big/ \sum_{t'=1}^{T} \frac{X_{it'}}{\hat{\pi}(Z_{it'})} & \text{if} \quad X_{it} = 1 \\ \frac{\left\{\sum_{t'=1}^{T} (1-X_{it'})\right\} Y_{it}}{1-\hat{\pi}(Z_{it})} \Big/ \sum_{t'=1}^{T} \frac{(1-X_{it'})}{1-\pi(Z_{it'})} & \text{if} \quad X_{it} = 0 \end{cases}$$

and the weights are the same as before

$$W_{it} \;\equiv\; \begin{cases} \frac{T}{\sum_{t'=1}^{T} X_{it'}} & \text{if} \quad X_{it} = 1, \\ \frac{T}{\sum_{t'=1}^{T} (1-X_{it'})} & \text{if} \quad X_{it} = 0. \end{cases}$$

Fast Computation and Standard Error Calculation

- Standard FE estimator:
  - "demean" both *Y* and *X*
  - regress demeaned *Y* on demeaned *X*
- Weighted FE estimator:
  - "weighted-demean" both *Y* and *X*
  - regress weighted-demeaned *Y* on weighted-demeaned *X*
- Model-based standard error calculation
  - Various robust sandwich estimators
  - Easy standard error calculation for matching estimators

Specification Test

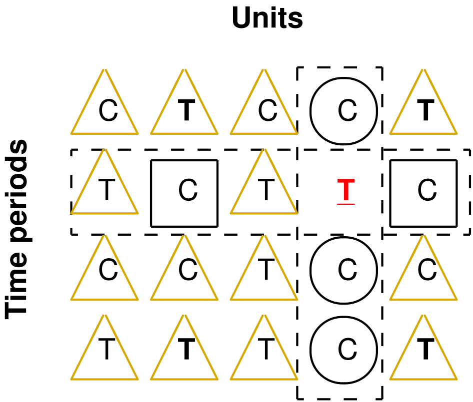
- Should we use standard or weighted FE models?

- Standard FE estimator is more efficient if its assumption is correct
- Weighted FE estimator is consistent under the same assumption
- White (1980) shows that any misspecified least squares estimator converges to the weighted least squares that minimizes mean squared prediction error

- Specification test:
  - Null hypothesis: standard FE model is correct
  - Does the difference between standard and weighted FE estimators arise by chance?

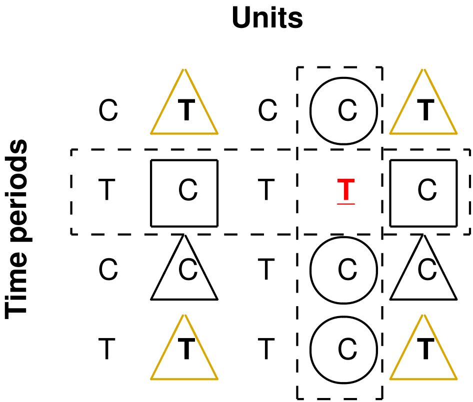
Mismatches in Two-Way FE Model

$$Y_{it} = \alpha_i + \gamma_t + \beta X_{it} + \epsilon_{it}$$



- **Triangles**: Two kinds of mismatches
  - Same treatment status
  - Neither same unit nor same time

Mismatches in Weighted Two-Way FE Model



- Some mismatches can be eliminated
- You can NEVER eliminate them all

Weighted Two-Way FE Estimator

Proposition 4

The adjusted matching estimator

$$\hat{\beta}^{M*} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{K_{it}}\left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)}\right)$$

$$\widehat{Y_{it}(x)} = \begin{cases} Y_{it} & \text{if } X_{it} = x \\ \frac{1}{m_{it}}\sum_{(i,t')\in\mathcal{M}_{it}} Y_{it'} + \frac{1}{n_{it}}\sum_{(i',t)\in\mathcal{N}_{it}} Y_{i't} - \frac{1}{m_{it}n_{it}}\sum_{(i',t')\in\mathcal{A}_{it}} Y_{i't'} & \text{if } X_{it} = 1-x \end{cases}$$

$$\mathcal{A}_{it} = \{(i',t') : i' \neq i, t' \neq t, X_{it'} = 1 - X_{it}, X_{i't} = 1 - X_{it}\}$$

$$K_{it} = \frac{m_{it}n_{it}}{m_{it}n_{it} + a_{it}}$$

and $m_{it} = |\mathcal{M}_{it}|$, $n_{it} = |\mathcal{N}_{it}|$, and $a_{it} = |\mathcal{A}_{it} \bigcap \{(i',t') : X_{i't'} = X_{it}\}|$.

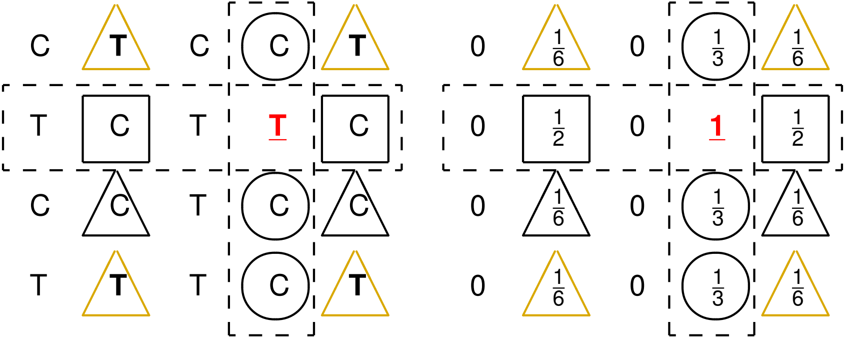
is equivalent to the following weighted two-way fixed effects estimator,

$$(\hat{\alpha}^{M*}, \hat{\gamma}^{M*}, \hat{\beta}^{M*}) = \underset{(\alpha,\beta,\gamma)}{\arg\min}\sum_{i=1}^{N}\sum_{t=1}^{T} W_{it}(Y_{it} - \alpha_i - \gamma_t - \beta X_{it})^2$$

Weighted Two-way Fixed Effects Model

$$\hat{\beta}^{M*} \;=\; \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\frac{1}{K_{it}}\left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)}\right)$$

Treatment **Weights**



General Difference-in-Differences Estimator is Equivalent to Weighted Two-Way FE Estimator

- Multiple time periods, repeated treatments

Treatment

C	**T**	C	C	**T**
T	C	T	**T**	C
C	C	T	C	C
T	**T**	T	C	**T**

Weights

0	0	0	0	0
0	$\frac{1}{2}$	0	**1**	$\frac{1}{2}$
0	$\frac{1}{2}$	0	1	$\frac{1}{2}$
0	0	0	0	0

- Difference-in-differences $=$ matching $=$ weighted two-way FE

Concluding Remarks

- Standard one-way and two-way FE estimators are adjusted matching estimators
- FE models are not a magic bullet solution to endogeneity
- In many cases, adjustment is not sufficient for removing bias

- Key Question: "Where are the counterfactuals coming from?"
- Different causal assumptions yield different weighted FE regressions
- Weighted FE encompasses a large class of causal assumptions: stratification, first differencing, propensity score weighting, difference-in-differences
- Model-based standard error, specification test

- Preliminary version of easyt-to-use software, R package **wfe**, available

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More information about this and other research:

http://imai.princeton.edu