Understanding and Improving Linear Fixed Effects Regression Models for Causal Inference

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EGAP Conference
November 3, 2011

Motivation

- Fixed effects models are a primary workhorse for causal inference in applied panel data analysis
- Researchers use them to adjust for unobservables:
  - “Good instruments are hard to find ..., so we’d like to have other tools to deal with unobserved confounders. This chapter considers ... strategies that use data with a time or cohort dimension to control for unobserved but fixed omitted variables” (Angrist & Pischke, *Mostly Harmless Econometrics*)
  - “fixed effects regression can scarcely be faulted for being the bearer of bad tidings” (Green *et al.*, *Dirty Pool*)
- Fixed effects models are often said to be superior to matching estimators because the latter can only adjust for observables
- **Question:** What are the exact causal assumptions underlying linear fixed effects regression models?

Main Results

1. Standard (one-way and two-way) linear fixed effects estimators are equivalent to particular matching estimators
2. Common belief that fixed effects models adjust for unobservables but matching does not is wrong
3. Identify the information used implicitly to estimate counterfactual outcomes under fixed effects models
4. Point out potential sources of bias and inefficiency in fixed effects estimators
5. Propose simple ways to improve fixed effects estimators using weighted linear fixed effects regression
6. Within-unit matching, first differencing, propensity score weighting, difference-in-differences are all weighted linear fixed effects with different regression weights

Matching and Regression in Cross-Section Settings

Units	**1**	**2**	**3**	**4**	**5**
Treatment status	**T**	**T**	**C**	**C**	**T**
Outcome	Y_1	Y_2	Y_3	Y_4	Y_5

- Estimating the Average Treatment Effect via matching

$$
\begin{aligned}
Y_1 &\quad - \quad \frac{1}{2}(Y_3 + Y_4) \\
Y_2 &\quad - \quad \frac{1}{2}(Y_3 + Y_4) \\
\frac{1}{3}(Y_1 + Y_2 + Y_5) &\quad - \quad Y_3 \\
\frac{1}{3}(Y_1 + Y_2 + Y_5) &\quad - \quad Y_4 \\
Y_5 &\quad - \quad \frac{1}{2}(Y_3 + Y_4)
\end{aligned}
$$

Matching Representation of Simple Regression

- Cross-section simple linear regression model:

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

- Binary treatment: $X_i \in \{0, 1\}$
- Equivalent matching estimator:

$$\hat{\beta} = \frac{1}{N} \sum_{i=1}^{N} \left(\widehat{Y_i(1)} - \widehat{Y_i(0)} \right)$$

where

$$\widehat{Y_i(1)} = \begin{cases} Y_i & \text{if } X_i = 1 \\ \frac{1}{\sum_{i'=1}^N X_{i'}} \sum_{i'=1}^N X_{i'} Y_{i'} & \text{if } X_i = 0 \end{cases}$$

$$\widehat{Y_i(0)} = \begin{cases} \frac{1}{\sum_{i'=1}^N (1-X_{i'})} \sum_{i'=1}^N (1 - X_{i'}) Y_{i'} & \text{if } X_i = 1 \\ Y_i & \text{if } X_i = 0 \end{cases}$$

- Treated units matched with the average of non-treated units

Fixed Effects Regression

- Simple (one-way) fixed effects regression:

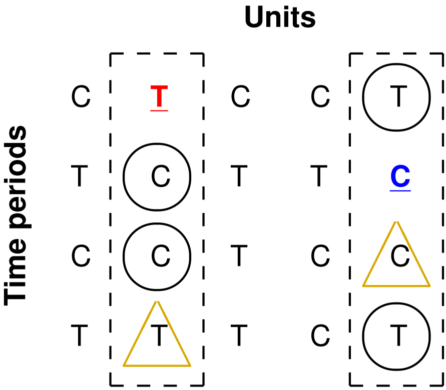
$$Y_{it} = \alpha_i + \beta X_{it} + \epsilon_{it}$$

- This estimator is in general inconsistent for the average treatment effect even if X_{it} is exogenous within each unit
- Instead, it converges to the weighted avearge of ATEs:

$$\hat{\beta}^{FE} \xrightarrow{p} \frac{\sum_{i=1}^{N} \mathbb{E}(Y_{it}(1) - Y_{it}(0)) \Pr(X_{it} = 1)\{1 - \Pr(X_{it} = 1)\}}{\sum_{i=1}^{N} \Pr(X_{it} = 1)\{1 - \Pr(X_{it} = 1)\}}$$

- Unit fixed effects $\Longrightarrow$ within-unit comparison
- Estimates of all counterfactual outcomes come from other time periods within the same unit
- How is this done under the fixed effects model?

Mismatches in One-way Fixed Effects Model



- T: treated observations
- C: control observations
- **Circles**: Proper matches
- **Triangles**: "Mismatches" $\Longrightarrow$ attenuation bias

Matching Representation of Fixed Effects Regression

Proposition 1

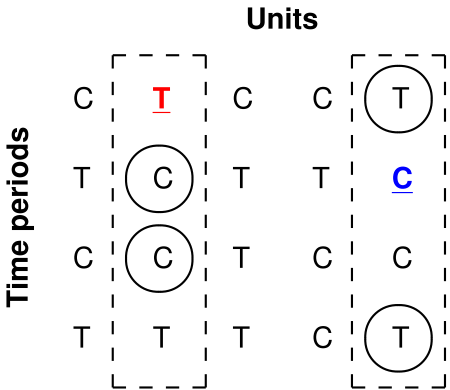
$$\hat{\beta}^{FE} = \frac{1}{K}\left\{\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)}\right)\right\},$$

$$\widehat{Y_{it}(x)} = \begin{cases} Y_{it} & \text{if } X_{it} = x \\ \frac{1}{T-1}\sum_{t' \neq t} Y_{it'} & \text{if } X_{it} = 1 - x \end{cases} \quad \textit{for} \quad x = 0, 1$$

$$K = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\{X_{it} \cdot \frac{1}{T-1}\sum_{t' \neq t}(1 - X_{it'}) + (1 - X_{it}) \cdot \frac{1}{T-1}\sum_{t' \neq t} X_{it'}\right\}.$$

- K: average proportion of proper matches across all observations
- More mismatches $\Longrightarrow$ larger adjustment
- Adjustment is required except very special cases
- “Fixes” attenuation bias but this adjustment is not sufficient
- Fixed effects estimator is a special case of matching estimators

Unadjusted Matching Estimator



- Consistent if the treatment is exogenous within each unit
- Only equal to fixed effects estimator if heterogeneity in either treatment assignment or treatment effect is non-existent

Unadjusted Matching as **Weighted** FE Estimator

Proposition 2

The unadjusted matching estimator

$$\hat{\beta}^M = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)}\right)$$

where

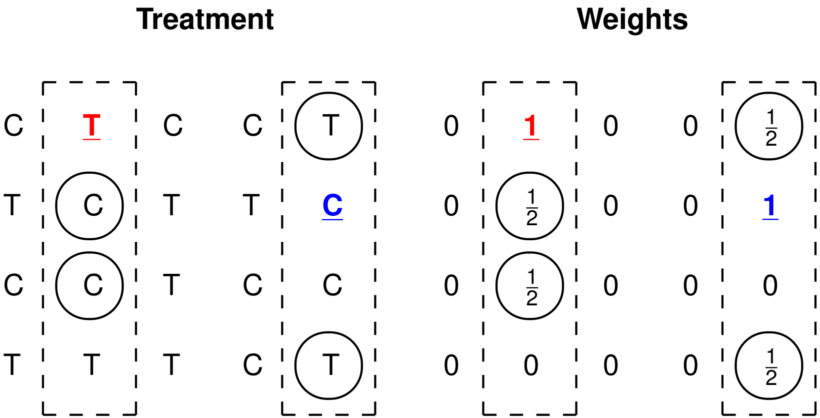
$$\widehat{Y_{it}(1)} = \begin{cases} Y_{it} & \text{if } X_{it} = 1 \\ \frac{\sum_{t'=1}^{T} X_{it'} Y_{it'}}{\sum_{t'=1}^{T} X_{it'}} & \text{if } X_{it} = 0 \end{cases} \quad \text{and} \quad \widehat{Y_{it}(0)} = \begin{cases} \frac{\sum_{t'=1}^{T}(1-X_{it'})Y_{it'}}{\sum_{t'=1}^{T}(1-X_{it'})} & \text{if } X_{it} = 1 \\ Y_{it} & \text{if } X_{it} = 0 \end{cases}$$

is equivalent to the weighted fixed effects model

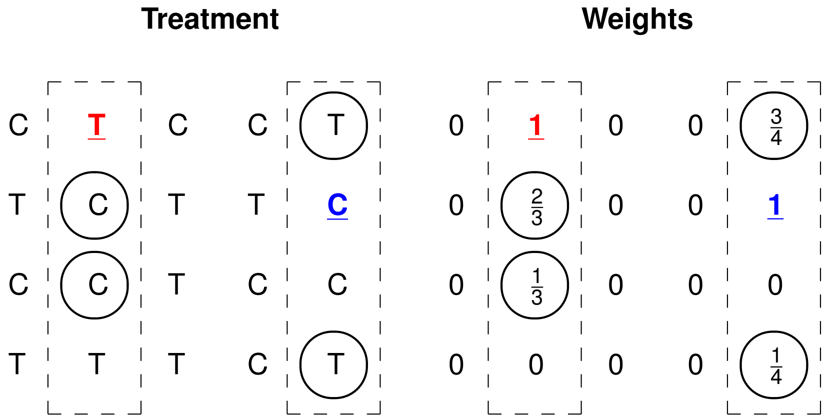
$$(\hat{\alpha}^M, \hat{\beta}^M) = \underset{(\alpha,\beta)}{\arg\min} \sum_{i=1}^{N}\sum_{t=1}^{T} W_{it}(Y_{it} - \alpha_i - \beta X_{it})^2$$

$$W_{it} \equiv \begin{cases} \frac{T}{\sum_{t'=1}^{T} X_{it'}} & \text{if} \quad X_{it} = 1, \\ \frac{T}{\sum_{t'=1}^{T}(1-X_{it'})} & \text{if} \quad X_{it} = 0. \end{cases}$$

Equal Weights



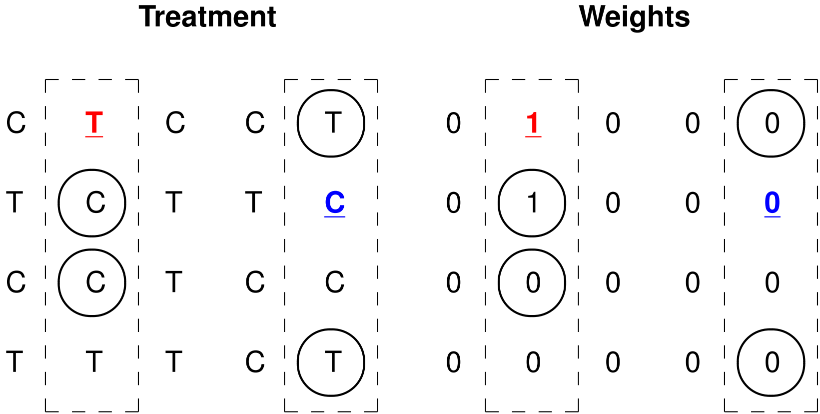
Different Weights



- Any within-unit matching procedure leads to weighted fixed effects regression with particular weights
- Theorem 1 shows how to derive regression weights given a matching procedure

First Differencing

- $\Delta Y_{it} = \beta \Delta X_{it} + \epsilon_{it}$ where $\Delta Y_{it} = Y_{it} - Y_{i,t-1}$, $\Delta X_{it} = X_{it} - X_{i,t-1}$



- First-difference = matching = weighted one-way fixed effects

Adjusting for Time-varying Observed Confounders

- Confounders Z_{it} are correlated with treatment and outcome

1. Regression-adjusted matching: $Y_{it} - \widehat{g(Z_{it})}$ where $g(z) = \mathbb{E}(Y_{it} \mid X_{it} = 0, Z_{it} = z)$
2. Linear regression adjustment with:

$$\underset{(\alpha,\beta,\delta)}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} W_{it}(Y_{it} - \alpha_i - \beta X_{it} - \delta^\top Z_{it})^2$$

   - *Ex post* interpretation: $Y_{it} - \hat{\delta}^\top Z_{it} \; = \; \alpha_i + \beta X_{it} + \epsilon_{it}$

3. Inverse-propensity score weighting with normalized weights

$$\hat{\beta}^W \quad = \quad \frac{1}{N} \sum_{i=1}^{N} \left\{ \sum_{t=1}^{T} \frac{X_{it} Y_{it}}{\hat{\pi}(Z_{it})} \Big/ \sum_{t=1}^{T} \frac{X_{it}}{\hat{\pi}(Z_{it})} - \sum_{t=1}^{T} \frac{(1 - X_{it}) Y_{it}}{1 - \hat{\pi}(Z_{it})} \Big/ \sum_{t=1}^{T} \frac{(1 - X_{it})}{1 - \hat{\pi}(Z_{it})} \right\}$$

where $\pi(Z_{it}) = \Pr(X_{it} = 1 \mid Z_{it})$ is the propensity score

   - within-unit weighting followed by across-units averaging

Propensity Score Weighting Estimator is Equivalent to Transformed Weighted FE Estimator

Proposition 3

$$(\hat{\alpha}^W, \hat{\beta}^W) = \underset{(\alpha,\beta)}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} W_{it}(Y_{it}^* - \alpha_i - \beta X_{it})^2$$

where the transformed outcome Y_{it}^* is,

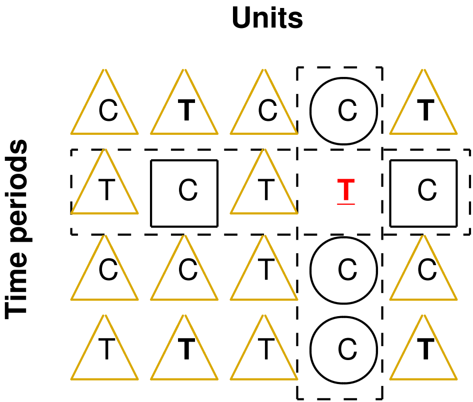
$$Y_{it}^* = \begin{cases} \frac{\left(\sum_{t'=1}^{T} X_{it'}\right) Y_{it}}{\hat{\pi}(Z_{it})} \Big/ \sum_{t'=1}^{T} \frac{X_{it'}}{\hat{\pi}(Z_{it'})} & \text{if} \quad X_{it} = 1 \\ \frac{\left\{\sum_{t'=1}^{T} (1-X_{it'})\right\} Y_{it}}{1-\hat{\pi}(Z_{it})} \Big/ \sum_{t'=1}^{T} \frac{(1-X_{it'})}{1-\pi(Z_{it'})} & \text{if} \quad X_{it} = 0 \end{cases}$$

and the weights are the same as before

$$W_{it} \equiv \begin{cases} \frac{T}{\sum_{t'=1}^{T} X_{it'}} & \text{if} \quad X_{it} = 1, \\ \frac{T}{\sum_{t'=1}^{T} (1-X_{it'})} & \text{if} \quad X_{it} = 0. \end{cases}$$

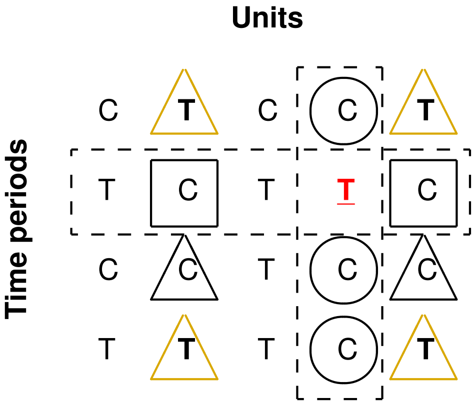
Mismatches in Two-way FE Model

$$Y_{it} = \alpha_i + \gamma_t + \beta X_{it} + \epsilon_{it}$$



- **Triangles**: Two kinds of mismatches
  - Same treatment status
  - Neither same unit nor same time

Mismatches in Weighted Two-way FE Model



- Some mismatches can be eliminated
- You can NEVER eliminate them all

Weighted Two-way FE Estimator

Proposition 4

The adjusted matching estimator

$$\hat{\beta}^{M*} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{1}{K_{it}} \left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)} \right)$$

$$\widehat{Y_{it}(x)} = \begin{cases} Y_{it} & \text{if } X_{it} = x \\ \frac{1}{m_{it}} \sum_{(i,t') \in \mathcal{M}_{it}} Y_{it'} + \frac{1}{n_{it}} \sum_{(i',t) \in \mathcal{N}_{it}} Y_{i't} - \frac{1}{m_{it} n_{it}} \sum_{(i',t') \in \mathcal{A}_{it}} Y_{i't'} & \text{if } X_{it} = 1 - x \end{cases}$$

$$\mathcal{A}_{it} = \{(i', t') : i' \neq i, t' \neq t, X_{it'} = 1 - X_{it}, X_{i't} = 1 - X_{it}\}$$

$$K_{it} = \frac{m_{it} n_{it}}{m_{it} n_{it} + a_{it}}$$

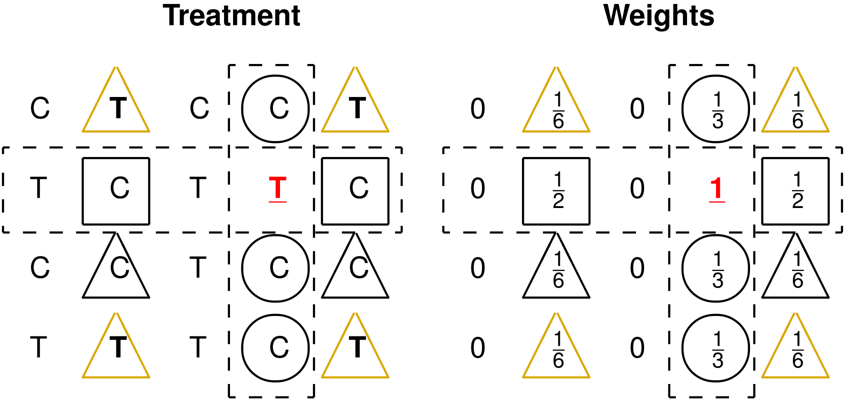
and $m_{it} = |\mathcal{M}_{it}|$, $n_{it} = |\mathcal{N}_{it}|$, and $a_{it} = |\mathcal{A}_{it} \bigcap \{(i', t') : X_{i't'} = X_{it}\}|$.

is equivalent to the following weighted two-way fixed effects estimator,

$$(\hat{\alpha}^{M*}, \hat{\gamma}^{M*}, \hat{\beta}^{M*}) = \underset{(\alpha,\beta,\gamma)}{\arg\min} \sum_{i=1}^{N} \sum_{t=1}^{T} W_{it} (Y_{it} - \alpha_i - \gamma_t - \beta X_{it})^2$$

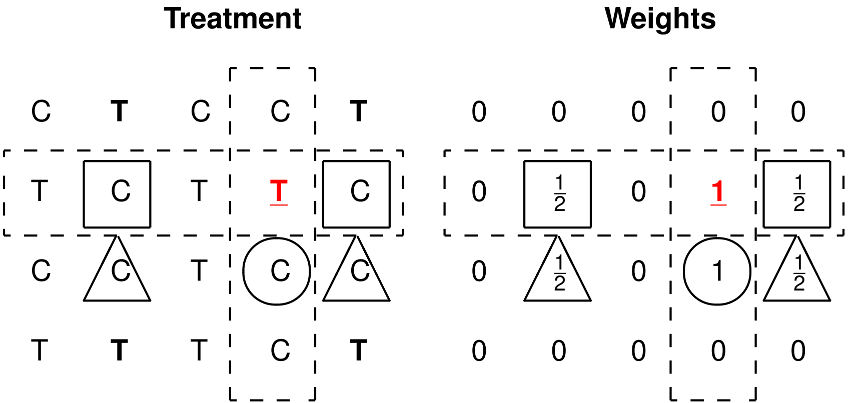
Weighted Two-way Fixed Effects Model

$$\hat{\beta}^{M*} = \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \frac{1}{K_{it}} \left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)} \right)$$



General Difference-in-Differences Estimator is Equivalent to Weighted Two-Way FE Estimator

- Multiple time periods, repeated treatments



- Difference-in-differences $=$ matching $=$ weighted two-way FE

Concluding Remarks and Practical Suggestions

- Standard one-way and two-way FE estimators are adjusted matching estimators
- FE models are not a magic bullet solution to endogeneity
- Key Question: “Where are the counterfactuals coming from?”
- Results can be sensitive to the underlying causal assumptions
- Different assumptions lead to different FE regression weights

- Our results show how to construct FE regression weights under a broad class of causal assumptions
- Within-unit matching, first differencing, propensity score weighting are all equivalent to weighted one-way FE estimators
- Difference-in-differences estimator is equivalent to the weighted two-way FE estimator

Theorem: General Equivalence between Weighted Fixed Effects and Matching Estimators

General matching estimator

$$\tilde{\beta}^M = \frac{1}{\sum_{i=1}^N \sum_{t=1}^T C_{it}} \sum_{i=1}^N \sum_{t=1}^T C_{it} \left(\widehat{Y_{it}(1)} - \widehat{Y_{it}(0)} \right)$$

where $0 \leq C_{it} < \infty$, $\sum_{t=1}^T \sum_{i=1}^N C_{it} > 0$,

$$\widehat{Y_{it}(1)} = \begin{cases} Y_{it} & \text{if } X_{it} = 1 \\ \sum_{t'=1}^T v_{it}^{it'} X_{it'} Y_{it'} & \text{if } X_{it} = 0 \end{cases}$$

$$\widehat{Y_{it}(0)} = \begin{cases} \sum_{t'=1}^T v_{it}^{it'} (1 - X_{it'}) Y_{it'} & \text{if } X_{it} = 1 \\ Y_{it} & \text{if } X_{it} = 0 \end{cases}$$

$$\sum_{t'=1}^T v_{it}^{it'} X_{it'} = \sum_{t'=1}^T v_{it}^{it'} (1 - X_{it'}) = 1$$

is equivalent to the weighted one-way fixed effects estimator

$$W_{it} = \sum_{i'=1}^N \sum_{t'=1}^T w_{it}^{i't'} \quad \text{and} \quad w_{it}^{i't'} = \begin{cases} C_{it} & \text{if } (i,t) = (i',t') \\ v_{it}^{it'} C_{i't'} & \text{if } (i,t) \in \mathcal{M}_{i't'} \\ 0 & \text{otherwise.} \end{cases}$$