Difference-in-Differences and Fixed Effects

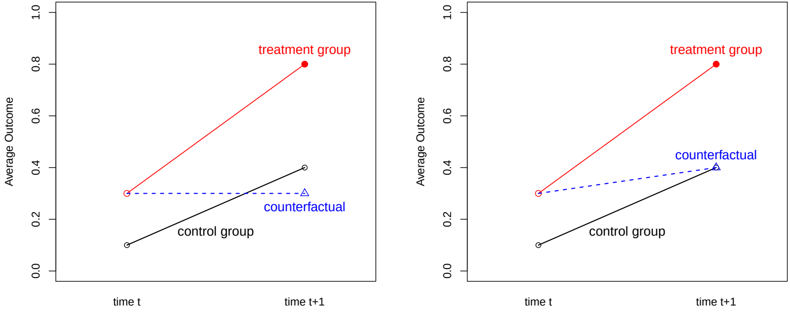
Kosuke Imai

Harvard University

Spring 2021

Motivation

- How should we conduct causal inference when repeated measurements are available?
- Two types of variations:
  1. cross-sectional variation within each time period
  2. temporal variation within each unit
- Before-and-after and cross-sectional designs



- Can we exploit both variations?

Minimum Wage and Unemployment

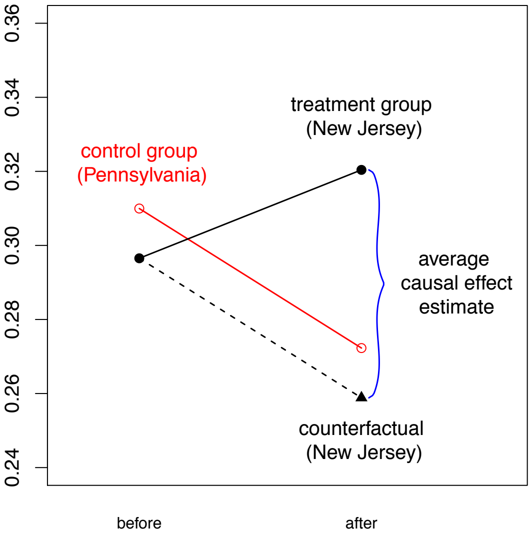
(Card and Krueger. 1994. *Am. Econ. Rev*)

- How does the increase in minimum wage affect employment?
- Many economists believe the effect is negative
  - especially for the poor
  - also for the whole economy
- Hard to randomize the minimum wage increase

- In 1992, NJ minimum wage increased from $4.25 to $5.05
  - Neighboring PA stays at $4.25
  - Observe employment in both states before and after increase
- NJ and (eastern) PA are similar
- Fast food chains in NJ and PA are similar: price, wages, products, etc.
- They are most likely to be affected by this increase

Difference-in-Differences Design

- Parallel trend assumption



- Setup:
  - Two time periods: time 0 (pre-treatment), time 1 (post-treatment)
  - G_i: treatment ($G_i = 1$) or control ($G_i = 0$) group
  - $Z_{it} = tG_i$: treatment assignment indicator for $t = 0, 1$
  - Potential outcomes: $Y_{i0}(0)$, $Y_{i0}(1)$, $Y_{i1}(0)$, $Y_{i1}(1)$
  - Observed outcomes: $Y_{it} = Y_{it}(Z_{it})$
- Average treatment effect for the treated:

$$\tau \;=\; \mathbb{E}\{Y_{i1}(1) - Y_{i1}(0) \mid G_i = 1\}$$

- Parallel trend assumption:

$$\mathbb{E}\{Y_{i1}(0) - Y_{i0}(0) \mid G_i = 1\} \;=\; \mathbb{E}\{Y_{i1}(0) - Y_{i0}(0) \mid G_i = 0\}$$

- DiD estimator:

$$\hat{\tau}_{\text{DiD}} \;=\; \underbrace{\{\widehat{\mathbb{E}(Y_{i1} \mid G_i} = 1) - \widehat{\mathbb{E}(Y_{i0} \mid G_i} = 1)\}}_{\text{difference for treated}} \\ -\underbrace{\{\widehat{\mathbb{E}(Y_{i1} \mid G_i} = 0) - \widehat{\mathbb{E}(Y_{i0} \mid G_i} = 0)\}}_{\text{difference for control}}$$

- Applicable to repeated cross-section data as well

Linear Model for the Difference-in-Differences

- Two-way fixed effects model:

$$Y_{it}(z) \;=\; \alpha_i + \beta t + \tau z + \epsilon_{it}$$

  - $\mathbb{E}\{Y_{i0}(0)\} \;=\; \alpha_i$
  - $\mathbb{E}\{Y_{i1}(0)\} \;=\; \alpha_i + \beta$
  - $\mathbb{E}\{Y_{i1}(1)\} \;=\; \alpha_i + \beta + \tau$
  - $\mathbb{E}\{Y_{i1}(1) - Y_{i1}(0)\} \;=\; \tau$

- Parallel trend assumption:
  - $\mathbb{E}\{Y_{i1}(0) - Y_{i0}(0) \mid G_i = g\} \;=\; \beta$
  - Or equivalently $\mathbb{E}(\epsilon_{i1} - \epsilon_{i0} \mid G_i = g) = 0$
  - Both Z_{it} and ϵ_{it} can depend on α_i or unobserved confounders

- Least squares estimator equals the nonparametric DiD estimator, i.e., $\hat{\tau}_{\text{FE}} = \hat{\tau}_{\text{DiD}}$
- This equivalence does not hold in general beyond the 2×2 case

Comparison with the Lagged Outcome Model

- Lagged outcome model:

$$Y_{i1}(z) \;=\; \alpha + \rho Y_{i0} + \tau z + \epsilon_i(z)$$

- Nonparametric identification assumption:

$$\{Y_{i1}(1), Y_{i1}(0)\} \perp\!\!\!\perp Z_{it} \mid Y_{i0}$$

  - can be made conditional on $\mathbf{X}_i$ as well as Y_{i0}
  - neither stronger nor weaker than the parallel trend assumption
  - same as parallel trend if $\mathbb{E}(Y_{i0} \mid G_i = 1) = \mathbb{E}(Y_{i0} \mid G_i = 0)$

- Where does the imbalance in lagged outcome come from?
  - Difference-in-Differences $\rightsquigarrow$ unobserved time-invariant confounder
  - Lagged outcome directly affects treatment assignment

Difference-in-Differences and Lagged Outcome Estimators

- Least squares estimator:

$$\hat{\tau}_{\text{LD}} = \underbrace{\widehat{\mathbb{E}(Y_{i1} \mid G_i = 1)} - \widehat{\mathbb{E}(Y_{i1} \mid G_i = 0)}}_{\text{difference for time 1}} - \hat{\rho}\{\underbrace{\widehat{\mathbb{E}(Y_{i0} \mid G_i = 1)} - \widehat{\mathbb{E}(Y_{i0} \mid G_i = 0)}}_{\text{difference for time 0}}\}$$

- If $\hat{\rho} = 1$, then $\hat{\tau}_{\text{LD}} = \hat{\tau}_{\text{DiD}}$
- Assume $0 \leq \rho < 1$ (stationarity)
- Without loss of generality, assume $\mathbb{E}(Y_{i0} \mid G_i = 1) \geq \mathbb{E}(Y_{i0} \mid G_i = 0)$ (monotonicity)
  1. If parallel trend holds, $\mathbb{E}(\hat{\tau}_{\text{LD}}) \geq \mathbb{E}(\hat{\tau}_{\text{DiD}}) = \tau$
  2. If ignorability holds, $\tau = \mathbb{E}(\hat{\tau}_{\text{LD}}) \geq \mathbb{E}(\hat{\tau}_{\text{DiD}})$
- Bracketing relationship: $\mathbb{E}(\hat{\tau}_{\text{LD}}) \geq \tau \geq \mathbb{E}(\hat{\tau}_{\text{DiD}})$
- Similar result holds nonparametrically (Ding and Li. 2019. *Political Anal.*)

Adjusting for Baseline Covariates

- Parallel trend assumption conditional on the baseline covaraites:

$$\mathbb{E}\{Y_{i1}(0) - Y_{i0}(0) \mid \mathbf{X}_i = \mathbf{x}, G_i = 1\}$$
$$= \mathbb{E}\{Y_{i1}(0) - Y_{i0}(0) \mid \mathbf{X}_i = \mathbf{x}, G_i = 0\} \quad \text{for all } \mathbf{x}$$

- Matching: parallel trend within a pair or a strata
- Weighting (Abadie. 2005. *Rev. Econ. Stud*):

$$\mathbb{E}\{Y_{i1}(1) - Y_{i1}(0) \mid G_i = 1\}$$
$$= \mathbb{E}\left[\frac{Y_{i1} - Y_{i0}}{\Pr(G_i = 1)} \cdot \frac{G_i - \Pr(G_i = 1 \mid \mathbf{X}_i)}{1 - \Pr(G_i = 1 \mid \mathbf{X}_i)}\right]$$

  where $\Pr(G_i = 1 \mid \mathbf{X}_i)$ is the propensity score

- Unconditional parallel trend assumption neither implies nor is implied by conditional parallel trend assumption

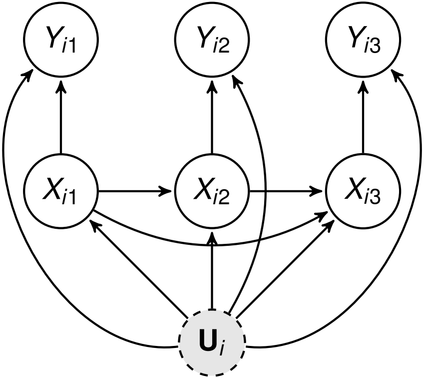
Fixed Effects Regression in Causal Inference

- Regression models with fixed effects are the primary workhorse for causal inference with panel data
- Researchers use them to adjust for unobserved time-invariant confounders (omitted variables, endogeneity, selection bias, ...)
  - "Good instruments are hard to find ..., so we'd like to have other tools to deal with unobserved confounders. This chapter considers ... strategies that use data with a time or cohort dimension to control for unobserved but fixed omitted variables" (Angrist & Pischke. 2009. *Mostly Harmless Econometrics*)
  - "fixed effects regression can scarcely be faulted for being the bearer of bad tidings" (Green *et al.* 2001. *Int. Organ.*)
- What are the causal assumptions of regressions with fixed effects?
- How are these models related to other causal inference methods?

Unit Fixed Effects Regression (Imai and Kim. 2019. *Am. J. Political Sci*)

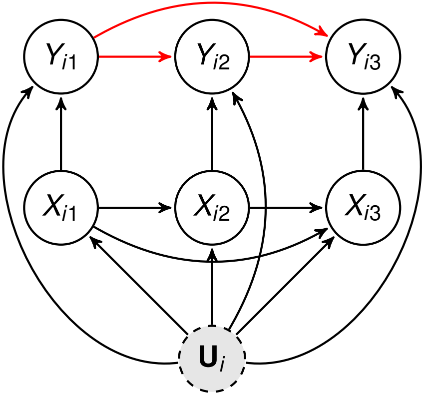
- One-way fixed effects linear regression: $Y_{it} = \alpha_i + \beta X_{it} + \epsilon_{it}$
- Strict exogeneity: $\mathbb{E}(\epsilon_{it} \mid \mathbf{X}_i, \alpha_i) = 0$
- Nonparametric structural equation model:

$$
\begin{aligned}
Y_{it} &= g_1(X_{it}, \mathbf{U}_i, \epsilon_{it}) \\
X_{it} &= g_2(X_{i1}, \ldots, X_{i,t-1}, \mathbf{U}_i, \eta_{it})
\end{aligned}
$$



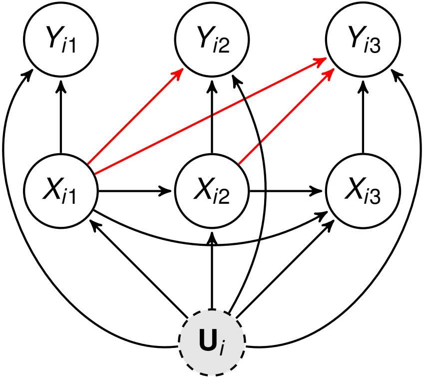
1. past treatments do not affect the current outcome
2. past outcomes do not affect the current outcome
3. past outcomes do not affect the current treatment

Past Outcomes Directly Affect Current Outcome



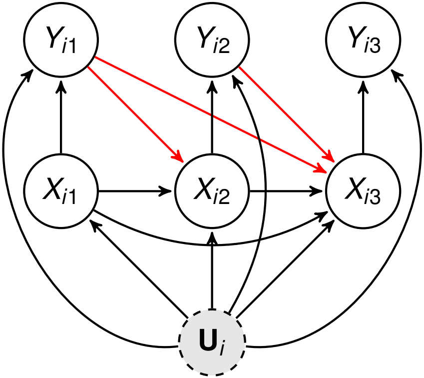
- Identification is still possible
- Past outcomes do not confound $X_{it} \longrightarrow Y_{it}$ given $\mathbf{U}_i$
- No need to adjust for past outcomes

Past Treatments Directly Affect Current Outcome



- Past treatments as confounders to be adjusted
- Strict exogeneity holds given past treatments and $\mathbf{U}_i$
- Impossible to adjust for an entire treatment history and $\mathbf{U}_i$ at the same time
- Adjust for a small number of past treatments $\rightsquigarrow$ often arbitrary

Past Outcomes Directly Affect Current Treatment



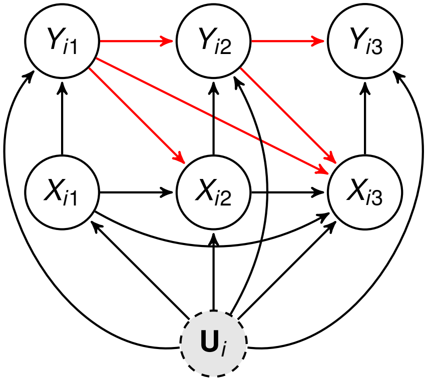
- Correlation between error term and future treatments
- Violation of strict exogeneity

- No adjustment is sufficient
- Together with the previous assumption
  $\rightsquigarrow$ no feedback effect over time

Instrumental Variables Approach

- AR(1) model with fixed effects:

$$Y_{it} = \alpha_i + \rho Y_{i,t-1} + \beta X_{it} + \epsilon_{it} \quad \text{where} \quad |\rho| < 1$$



- Instruments: X_{i1}, X_{i2}, and Y_{i1}
- Generalized Method of Moments (GMM): Arellano and Bond (1991)
- Exclusion restrictions
- Arbitrary choice of instruments
- Substantive justification rarely given