Inference for Average Treatment Effects

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Agenda

- Average treatment effects as causal quantities of interest:
  1. Sample Average Treatment Effect (SATE)
  2. Population Average Treatment Effect (PATE)

- Difference-in-means estimator

- Design-based approach:
  - randomization of treatment assignment,
  - random sampling

- Statistical inference:
  - exact moments
  - asymptotic confidence intervals

Social Pressure and Turnout (Gerber, et al. 2008. *Am. Political Sci. Rev.*)

- August 2006 Primary Election in Michigan
- 180,000 households
- Send postcards with different messages

- Randomly assign each household to a group (or treatment)
  1. no message (control group)
  2. civic duty message
  3. "you are being studied" message (Hawthorne effect)
  4. neighborhood social pressure message

Neighborhood Social Pressure Message

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted i the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

“You are being studied” Message

Dear Registered Voter:

YOU ARE BEING STUDIED!

Why do so many people fail to vote? We've been talking about this problem for years, but it only seems to get worse.

This year, we're trying to figure out why people do or do not vote. We'll be studying voter turnout in the August 8 primary election.

Our analysis will be based on public records, so you will not be contacted again or disturbed in any way. Anything we learn about your voting or not voting will remain confidential and will not be disclosed to anyone else.

DO YOUR CIVIC DUTY — VOTE!

Standard Empirical Analysis

Groups	Control	Civic duty	Hawthorne	Neighbor
Turnout rate	29.7%	31.5%	32.2%	37.5%
# of voters	191,243	38,218	38,204	38,201

- Neighborhood social pressure vs. Control

$$\begin{aligned}
\hat{\tau} &= \frac{1}{n_1}\sum_{i=1}^{n} T_i Y_i - \frac{1}{n_0}\sum_{i=1}^{n}(1-T_i)Y_i \approx 7.8 \\
s.e. &= \sqrt{\frac{\widehat{\mathbb{V}(Y_i \mid T_i = 1)}}{n_1} + \frac{\widehat{\mathbb{V}(Y_i \mid T_i = 0)}}{n_0}} \approx 0.3 \\
95\%\mathrm{CI} &= [\hat{\tau} - z_{0.975} \times s.e., \ \hat{\tau} + z_{0.975} \times s.e.] \approx [7.2, \ 8.4]
\end{aligned}$$

- How can we justify this standard difference-in-means analysis from the randomization perspective?

Estimation of the Sample Average Treatment Effect

- Due to Neyman (1923) (Neyman. 1990 (translated to English) *Stat. Sci.*)
- Difference-in-means estimator:

$$\hat{\tau} \;=\; \frac{1}{n_1}\sum_{i=1}^{n} T_i Y_i - \frac{1}{n_0}\sum_{i=1}^{n}(1 - T_i) Y_i$$

- Unbiasedness (over repeated treatment assignments):

$$\begin{aligned}
\mathbb{E}(\hat{\tau} \mid \mathcal{O}_n) \;&=\; \frac{1}{n_1}\sum_{i=1}^{n} \mathbb{E}(T_i \mid \mathcal{O}_n) Y_i(1) - \frac{1}{n_0}\sum_{i=1}^{n}\{1 - \mathbb{E}(T_i \mid \mathcal{O}_n)\} Y_i(0) \\
&=\; \frac{1}{n}\sum_{i=1}^{n}(Y_i(1) - Y_i(0)) \;=\; \text{SATE}
\end{aligned}$$

where $\mathcal{O}_n = \{Y_i(0), Y_i(1)\}_{i=1}^{n}$ and $\mathbb{E}(T_i \mid \mathcal{O}_n) = n_1/n$ for all i

The Variance of the Difference-in-Means Estimator

- Randomness comes only from the treatment assignment
- Variance of $\hat{\tau}$:

$$\mathbb{V}(\hat{\tau} \mid \mathcal{O}_n) = \frac{1}{n}\left(\frac{n_0}{n_1} S_1^2 + \frac{n_1}{n_0} S_0^2 + 2 S_{01}\right),$$

where for $t = 0, 1$,

$$S_t^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i(t) - \overline{Y(t)})^2 \quad \text{sample variance of } Y_i(t)$$

$$S_{01} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i(0) - \overline{Y(0)})(Y_i(1) - \overline{Y(1)}) \quad \text{sample covariance}$$

- The variance is NOT *identifiable*

Bounds on the Variance

- Cauchy-Schwartz inequality:

$$-S_1 S_0 \ \leq \ S_{01} \ \leq \ S_1 S_0$$

  1. Upper bound: $\mathrm{corr}(Y_i(1), Y_i(0)) = 1$
  2. Lower bound: $\mathrm{corr}(Y_i(1), Y_i(0)) = -1$

$$\frac{n_0 n_1}{n}\left(\frac{S_1}{n_1} - \frac{S_0}{n_0}\right)^2 \ \leq \ \mathbb{V}(\hat{\tau} \mid \mathcal{O}_n) \ \leq \ \frac{n_0 n_1}{n}\left(\frac{S_1}{n_1} + \frac{S_0}{n_0}\right)^2$$

- Sharp bounds based on the entire marginal distributions (Aronow et al. 2015. *Ann. Stat.*)

Conservative Variance Estimator

- The usual variance estimator is conservative on average:

$$\mathbb{V}(\hat{\tau} \mid \mathcal{O}_n) \quad \leq \quad \frac{S_1^2}{n_1} + \frac{S_0^2}{n_0} \;=\; \mathbb{E}\left(\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_0^2}{n_0} \,\middle|\, \mathcal{O}_n\right)$$

  where

$$\hat{\sigma}_t^2 \;=\; \frac{1}{n_t - 1}\sum_{i=1}^{n} \mathbf{1}\{T_i = t\}(Y_i - \overline{Y}_t)^2 \quad \text{for } t = 0, 1$$

- Under the constant additive unit causal effect assumption, i.e., $Y_i(1) - Y_i(0) = c$ for all i, we have $\text{var}(Y_i(1) - Y_i(0)) = 0$, implying

$$S_{01} \;=\; \frac{1}{2}(S_1^2 + S_0^2) \quad \text{and} \quad \mathbb{V}(\hat{\tau} \mid \mathcal{O}_n) \;=\; \frac{S_1^2}{n_1} + \frac{S_0^2}{n_0}$$

- The optimal treatment assignment rule:

$$n_1^{opt} = \frac{n}{1 + S_0/S_1}, \qquad n_0^{opt} = \frac{n}{1 + S_1/S_0}$$

Inference for Population Average Treatment Effect

- Assumption: simple random sampling from a super-population
- Unbiasedness (over repeated sampling):

$$\mathbb{E}(\hat{\tau}) \;=\; \mathbb{E}\{\mathbb{E}(\hat{\tau} \mid \mathcal{O}_n)\} \;=\; \mathbb{E}(\mathsf{SATE}) \;=\; \mathbb{E}\{Y_i(1) - Y_i(0)\} \;=\; \mathsf{PATE}$$

- Variance:

$$\begin{aligned}\mathbb{V}(\hat{\tau}) &= \mathbb{V}\{\mathbb{E}(\hat{\tau} \mid \mathcal{O}_n)\} + \mathbb{E}\{\mathbb{V}(\hat{\tau} \mid \mathcal{O}_n)\} \\ &= \frac{\sigma_1^2}{n_1} + \frac{\sigma_0^2}{n_0}\end{aligned}$$

  where σ_t^2 is the population variance of $Y_i(t)$ for $t = 0, 1$
- Unbiased variance estimator:

$$\widehat{\mathbb{V}(\hat{\tau})} \;=\; \frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_0^2}{n_0} \quad \text{where} \quad \mathbb{E}\{\widehat{\mathbb{V}(\hat{\tau})}\} = \mathbb{V}(\hat{\tau})$$

  for $t = 0, 1$

Asymptotic Inference

- Hold $k = n_1/n$ constant and let n goes to infinity

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{T_i Y_i(1)}{k} - \frac{(1-T_i)Y_i(0)}{1-k}\right)$$

- PATE
  - Consistency via the law of large numbers: $\hat{\tau} \xrightarrow{p}$ PATE
  - Asymptotic normality via the Central Limit Theorem (CLT)

$$\frac{\hat{\tau} - \text{PATE}}{\sqrt{\sigma_1^2/n_1 + \sigma_0^2/n_0}} \xrightarrow{d} \mathcal{N}(0,1)$$

- SATE: finite population CLT (Li and Ding. 2017. *J. Am. Stat. Assoc.*)

$$\frac{\hat{\tau} - \mathrm{SATE}}{\sqrt{(n_0 S_1^2/n_1 + n_1 S_0^2/n_0 + 2S_{01})/n}} \xrightarrow{d} \mathcal{N}(0,1)$$

- $(1-\alpha)\times 100\%$ CI: $[\hat{\tau} - \mathrm{s.e.} \times z_{1-\alpha/2},\ \hat{\tau} + \mathrm{s.e.} \times z_{1-\alpha/2}]$

Summary: Fisher vs. Neyman

- Like Fisher, Neyman proposed randomization-based inference
- Two limitations of Fisher's permutation inference:
  1. causal heterogeneity
  2. population inference
- Unlike Fisher,
  1. estimands are *average* treatment effects
  2. heterogenous treatment effects are allowed
  3. population as well as sample inference is possible
  4. asymptotic approximation is required for inference

Exchange at the Royal Statistical Society

(Neyman *et al.* (1935) *Suppl. of J. Royal Stat. Soc.*)

Neyman: *So long as the average yields of any treatments are identical, the question as to whether these treatments affect separate yields on single plots seems to be uninteresting*

Fisher: *It may be foolish, but that is what the* z *test was designed for, and the only purpose for which it has been used.*

Neyman: *I am considering problems which are important from the point of view of agriculture.*

Fisher: *It may be that the question which Dr. Neyman thinks should be answered is more important than the one I have proposed and attempted to answer. I suggest that before criticizing previous work it is always wise to give enough study to the subject to understand its purpose.*