POL502: Limits of Functions and Continuity

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October 18, 2005

In this chapter, we study limits of functions and the concept of continuity. Now that we have a good understanding of limits of sequences, it should not be too difficult to investigate limits of functions. Indeed, as we shall see below, there exists a strong connection between sequential and functional limits. Throughout the chapter, we focus on real-valued functions, $f : X \mapsto \mathbf{R}$ where $X \subset \mathbf{R}$.

1 Limits of Functions

First, we formally define the limit of functions

Definition 1 *Let $f : X \mapsto \mathbf{R}$, and let c be an accumulation point of the domain X. Then, we say f has a limit L at c and write $\lim_{x \to c} f(x) = L$, if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - c| < \delta$ and $x \in X$ imply $|f(x) - L| < \epsilon$.*

A few remarks about this definition are worthwhile. First, $0 < |x - c|$ is an economical way of saying that x differs from c. Second, recall the definition of an accumulation point (or limit point) of a set. Every neighborhood of such a point must contain infinitely many points of the set that are different from itself. Also, recall that an accumulation point does not have to belong to the set. That is, c might not belong to X. Together, the definition says that we must find $\delta > 0$ such that if $x \in (c - \delta, c + \delta)$, then $f(x) \in (L - \epsilon, L + \epsilon)$. Note that the choice of δ often depends on the choice of ϵ. Finally, if f has a limit L at c, it is unique. You should be able to prove this fact by now.

Example 1 *Find the following limits if they exist.*

1. $f : \mathbf{R} \mapsto \mathbf{R}$ *and* $\lim_{x \to 2} f(x)$ *where* $f(x) = 2x + 1$.
2. $f : \mathbf{R} \mapsto \mathbf{R}$ *and* $\lim_{x \to 1} f(x)$ *where* $f(x) = \frac{x^2 - 1}{x - 1}$ *if* $x \neq 1$ *and* $f(x) = 6$ *if* $x = 1$.
3. $f : \mathbf{R} \setminus 0 \mapsto \mathbf{R}$ *and* $\lim_{x \to 0} f(x)$ *where* $f(x) = \frac{|x|}{x}$.
4. $f : (0, 1) \mapsto \mathbf{R}$ *and* $\lim_{x \to 0} f(x)$ *where* $f(x) = 1/x$.

In some cases, there is no finite limit but the function is approaching to infinity (or minus infinity). Then, we write $\lim_{x \to c} f(x) = \infty$ or $\lim_{x \to c} f(x) = -\infty$.

Definition 2 *Let $f : X \mapsto \mathbf{R}$ be a function and $c \in X$ be an accumulation point of X. $\lim_{x \to c} f(x) = \infty$ if for every $m > 0$ there exists $\delta > 0$ such that $f(x) \geq m$ for any x with $0 < |x - c| < \delta$.*

$\lim_{x\to c} f(x) = -\infty$ can be defined in a similar way.

In the above example, we saw that the function $1/x$ does not have a limit at zero because it "blows up" near zero. This fact can be generalized by the following theorem,

Theorem 1 (Limits and Bounds of Functions) *Let $f : X \mapsto \mathbf{R}$ and suppose c is an accumulation point of X. If f has a limit at c, then there is a neighborhood Q of c and a real number m such that for all $x \in Q \cap X$, $|f(x)| \leq m$.*

By now, you might guess that there is the strong connection between limits of sequences and functions. The next theorem directly establishes this connection. Once we prove it, we can apply to limits of functions many results that we have derived for limits of sequences. In fact, the previous theorem can also be proved by applying this theorem.

Theorem 2 (Sequential and Functional Limits) *Let $f : X \mapsto \mathbf{R}$, and let c be an accumulation point of X. Then, f has a limit L at c if and only if the sequence $\{f(x)\}_{n=1}^{\infty}$ converges to L for any sequence $\{x_n\}_{n=1}^{\infty}$ converging to c with $x_n \in X$ and $x_n \neq c$ for all n.*

The following results follow immediately from this theorem.

Theorem 3 (Algebraic Operations of Functional Limits) *Let $f, g : X \mapsto \mathbf{R}$ with c being an accumulation point of X. Assume that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$.*

1. *$\lim_{x\to c} kf(x) = kL$ for all $k \in \mathbf{R}$.*
2. *$\lim_{x\to c}\{f(x) + g(x)\} = L + M$.*
3. *$\lim_{x\to c}\{f(x)g(x)\} = LM$.*
4. *$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$.*

The functional limits also respect the order,

Theorem 4 (Order of Functional Limits) *Let $f, g : X \mapsto \mathbf{R}$ and c be an accumulation point of X. Assume that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. If $f(x) \leq g(x)$ for all $x \in X$, then $L \leq M$.*

Finally, monotone functions can be defined in a manner completely analogous to monotone sequences.

Definition 3 (Monotone Functions) *Let $f : X \mapsto \mathbf{R}$.*

1. *The function f is said to be increasing if $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \leq x_2$.*
2. *The function f is said to be decreasing if $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \leq x_2$.*
3. *If f is either increasing or decreasing, then f is said to be monotone.*

2 Continuous Functions

In the discussion of functional limits, we did not care at all about the value of f at c, i.e., $f(c)$. In fact, c, an accumulation point of X, did not have to belong to X. When discussing continuity on the other hand, we need to focus our attention on the value of f at c.

Definition 4 (Continuity) *Let $f : X \mapsto \mathbf{R}$ be a function.*

1. *f is said to be continuous at $c \in X$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ and $x \in X$ imply $|f(x) - f(c)| < \epsilon$.*

2. *f is said to be continuous on X if it is continuous at every $x \in X$.*

In words, f is continuous at c if $f(x)$ is close to $f(c)$ whenever x is sufficiently close to c. Let's compare this definition with the definition of functional limits. There is no requirement that c be an accumulation point. Instead, c must belong to the set X. Finally, we do not require x to be different from c. In fact, if c is not an accumulation point and c belongs to X, then f is automatically continuous at c because you can always find $\delta > 0$ small enough so that $|x - c| < \delta$ and $x \in X$ imply $x = c$ and hence $|f(x) - f(c)| = 0 < \epsilon$ for any $\epsilon > 0$. Therefore, the only interesting case is that c is an accumulation point and belongs to X. In this case, the definition of continuity requires that $f(c)$ to be the limit at c. Now, let's look at examples.

Example 2 *Are the following functions continuous?*

1. *$f : [0, \infty) \mapsto \mathbf{R}$ where $f(x) = \sqrt{x}$.*

2. *$f : \mathbf{R} \setminus 0 \mapsto \mathbf{R}$ where $f(x) = \frac{1}{x}$.*

We characterize the relationship between continuity and limit with the following theorem,

Theorem 5 (Characterization of Continuity) *Let $f : X \mapsto \mathbf{R}$ be a function and $c \in X$ be an accumulation point of X. Then, the following three statements are equivalent.*

1. *f is continuous at c.*

2. *$\lim_{x \to c} f(x) = f(c)$.*

3. *For every sequence $\{x_n\}_{n=1}^{\infty}$ converging to c with $x_n \in X$ for all n, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(c)$.*

As expected, usual algebraic operations work well with continuous functions.

Theorem 6 (Algebraic Operations of Continuous Functions) *Let $f : X \mapsto \mathbf{R}$ and $g : X \mapsto \mathbf{R}$ be continuous at $c \in X$. Then,*

1. *$f + g$ is continuous at c.*

2. *fg is continuous at c.*

3. *$\frac{f}{g}$ is continuous at c provided $g(c) \neq 0$.*

The following theorem allows one to expand the class of continuous functions easily.

Theorem 7 (Continuity of Compound Functions) *Let $f : X \mapsto \mathbf{R}$ and $g : Y \mapsto \mathbf{R}$ with $f(X) \subset Y$. Assume that f is continuous at $c \in X$ and g is continuous at $f(c)$. Then, $g \circ f$ is continuous at c.*

These two theorems are very powerful. For example, you can very easily prove $f : \mathbf{R} \mapsto \mathbf{R}$ where $f(x) = \sqrt{2x^2 + 1}$ is continuous.

3 Uniform Continuity and Compact Sets

In this section, we consider the concept of uniform continuity, which is formally defined as follows

Definition 5 *A function $f : X \mapsto \mathbf{R}$ is uniformly continuous on X if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ and $x, y \in X$ imply $|f(x) - f(y)| < \epsilon$.*

Compare this with the definition of continuity. Uniform continuity is stronger than continuity. That is, a function can be continuous even if it is not uniformly continuous, though every uniformly continuous function is also continuous. Continuity says that given $\epsilon > 0$ and a particular point $c \in X$ we can find a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$. The choice of δ may depend on c as well as ϵ. In contrast, uniform continuity says that for *all* $c \in X$ and any given ϵ, we can find a δ such that $|f(x) - f(c)| < \epsilon$. Here, the choice of δ cannot depend on c though it may depend on ϵ. To further help us distinguish these two concepts, consider the following examples.

Example 3 *Are the following functions uniformly continuous?*

1. $f : \mathbf{R} \mapsto \mathbf{R}$ *defined by* $f(x) = 3x + 1$.
2. $f : (2.5, 3) \mapsto \mathbf{R}$ *defined by* $f(x) = 3/(x - 2)$.
3. $f : \mathbf{R} \mapsto \mathbf{R}$ *defined by* $f(x) = x^2$.
4. $f : (0, 1) \mapsto \mathbf{R}$ *defined by* $f(x) = 1/x$.

The next theorem proves the connection between uniform continuity and limit.

Theorem 8 (Uniform Continuity and Limits) *Let $f : X \mapsto \mathbf{R}$ be a uniformly continuous function. If c is an accumulation point of X, then f has a limit at c.*

In order to further investigate the relationship between continuity and uniform continuity, we need to introduce some new concepts.

Definition 6 *Let $X \subset \mathbf{R}$ be a set.*

1. *X is an open set if for each $x \in X$ there exists a neighborhood Q of x such that $Q \subset X$.*
2. *X is a closed set if every accumulation point of X belongs to X.*

As we expect, an open interval is an open set and a closed interval is a closed set.

Example 4 *Let $a, b \in \mathbf{R}$. Then,*

1. $(a, b) = \{x : a < x < b\}$ *is open.*
2. $[a, b] = \{x : a \leq x \leq b\}$ *is closed.*

$[a, b]$ is an example of a *perfect set*, a closed set X such that every $x \in X$ is also an accumulation point of X. Now, we establish the relationship between open and closed sets.

Theorem 9 (Open and Closed Sets) *Let $X \subset \mathbf{R}$ be a set. X is closed if and only if $\mathbf{R} \setminus X$ is open.*

We define the notion of a compact set. Before doing so, we must introduce some terminology.

Definition 7 *Let $X \subset \mathbf{R}$ be a set and $\{Y_\alpha\}_{\alpha\in A}$ be a collection of sets.*

1. *$\{Y_\alpha\}_{\alpha\in A}$ is a cover of X if $X \subset \bigcup_{\alpha\in A} Y_\alpha$.*
2. *$\{Y_\alpha\}_{\alpha\in A}$ is an open cover of X if $\{Y_\alpha\}_{\alpha\in A}$ is a cover of X and each Y_α is an open set.*
3. *$\{Y_\alpha\}_{\alpha\in B}$ is a subcover of X if both $\{Y_\alpha\}_{\alpha\in A}$ and $\{Y_\alpha\}_{\alpha\in B}$ are a cover of X and $B \subset A$.*
4. *$\{Y_\alpha\}_{\alpha\in B}$ is a finite subcover of X if $\{Y_\alpha\}_{\alpha\in B}$ is a subcover of X and B is a finite set.*

Definition 8 *Let $X \subset \mathbf{R}$ be a set. X is compact if every open cover of X has a finite subcover.*

What a mess! We will try to understand this definition first by examples and then an important theorem about compactness.

Example 5 *Find an open cover for the following sets. Can you also find a finite subcover?*

1. $(0, 1)$.
2. $[0, 1]$.
3. $(0, 1]$.
4. $\mathbf{N}$.

Notice that $(0, 1)$ is open and bounded, $[0, 1]$ is closed and bounded, and $\mathbf{N}$ is closed and unbounded. Now, we can guess that compact sets must be closed and bounded. This intuition is confirmed by the following important theorem.

Theorem 10 (Heine-Borel) *Let $X \subset \mathbf{R}$ be a set. X is compact if and only if it is closed and bounded.*

Finally, we show that a continuous function with the domain being a compact set is uniformly continuous.

Theorem 11 (Continuity and Uniform Continuity) *Let $f : X \mapsto \mathbf{R}$ be a continuous function. If X is compact, then f is uniformly continuous.*

4 Properties of Continuous Functions

The results of the previous section has many important implications, which we further explore in this section. First, we show that uniformly continuous functions preserve bounded sets.

Theorem 12 (Preservation of Bounded Sets) *If $f : X \mapsto \mathbf{R}$ is uniformly continuous on a bounded set X, then $f(X)$ is a bounded set.*

Next, suppose $f : X \mapsto \mathbf{R}$ is continuous with X compact. Then, f is uniformly continuous and X is bounded. Therefore, $f(X)$ is also bounded. If we can prove that $f(X)$ is closed, then it implies that continuous functions preserve compact sets. Before we prove this, we need to establish a connection between compact sets and convergence of sequences by recalling some key properties of bounded sequences (i.e., Bolzano-Weirstrass Theorem).

Theorem 13 (Sequences and Compact Sets) *A set X is compact if and only if every sequence in X has a subsequence that converges to a limit $x \in X$.*

Next, we are ready to prove the theorem,

Theorem 14 (Preservation of Compact Sets) *Let $f : X \mapsto \mathbf{R}$ be a continuous function. If $Y \subset X$ is a compact set, then $f(Y)$ is also compact.*

Given this result, the following theorem is immediate,

Theorem 15 (Extreme Value Theorem) *If $f : X \mapsto \mathbf{R}$ is continuous on a compact set X, then there exists $x_0, x_1 \in X$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.*

In words, this theorem implies that a continuous function on a compact set is guaranteed to have a maximum and a minimum. This leads to the next theorem.

Theorem 16 (Intermediate Value Theorem) *Let $f : [x_0, x_1] \mapsto \mathbf{R}$ be a continuous function. If $f(x_0) < 0 < f(x_1)$ or $f(x_0) > 0 > f(x_1)$, then there exists $c \in (x_0, x_1)$ such that $f(c) = 0$.*

The intermediate theorem holds even if one replaces 0 with an arbitrary number $y \in \mathbf{R}$. One can easily prove this by considering a function $g(x) = f(x) - y$.

Now, we use the Intermediate Value theorem to prove a couple of useful theorems.

Theorem 17 (Root of Polynomial) *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be a polynomial of odd degree with real coefficients, i.e., $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ where n is odd. Then the equation $f(x) = 0$ has at least one real root.*

The story about the next theorem goes as follows. Consider two sheets of papers of identical size, one lying on top of the other. Now, you crumple the top sheet (be careful not to rip it) and put it back on top of the other sheet. It turns out that there exists at least one point on the top sheet which lies exactly above the corresponding point of the bottom sheet. Let's prove this.

Definition 9 *Let f be a real-valued function. x is a fixed point of f if $f(x) = x$.*

Theorem 18 (Fixed Point Theorem) *Let $f : [a, b] \mapsto [a, b]$ be continuous on $[a, b]$. Then, f has a fixed point in $[a, b]$.*

The proof was so easy, wasn't it? Do you feel empowered now? Armed with the solid understanding of functional limits and continuity, we now move on to the next topic, calculus.