POL502: Sequences

Kosuke Imai
Department of Politics, Princeton University

October 18, 2005

With the foundations we have built so far, we will study sequences.

1 Limits of Sequences

Definition 1 (Sequence) *Let X be a set. A sequence in X is a function from* $\mathbf{N}$ *to X.*

In words, a sequence is a function that takes an input from $\mathbf{N}$ and produces an output in X. There must be some pattern that can be described in a certain way.

Example 1 *Two examples of a sequence:*

1. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.
2. $\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}$.

The focus of our study is the behavior of these infinite sequences. We say a sequence is convergent if it eventually becomes *arbitrarily close* to some number after a certain point. We now formally define the concept of convergence.

Definition 2 (Limit of Sequence) *A sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit x if for all $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|a_n - x| < \epsilon$ for all $n \geq N$. Then, we write $\lim_{n\to\infty} a_n = x$ or more economically $\lim a_n = x$. A sequence that does not converge is said to be divergent.*

It might be helpful to think of $|x_n - x| < \epsilon$ as a set of points around x. We call such a set as "neighborhood of x." Note that ϵ represents a quantity that can be made arbitrarily small.

Definition 3 (Neighborhood) *A set Q is a neighborhood of a point x if there exists $\epsilon > 0$ such that the open interval $(x - \epsilon, x + \epsilon)$ is a subset of Q.*

Given this definition of convergence, let's look at some examples.

Example 2 *Find the limit of the following sequences.*

1. $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$
2. $\left\{\frac{n+1}{n}\right\}_{n=1}^{\infty}$
3. $\{\sqrt{n+1} - \sqrt{n}\}_{n=1}^{\infty}$

Now, we prove some important theorems about convergent sequences. The first such theorem is about the uniqueness of limit. This makes intuitive sense because a sequence cannot be arbitrarily close to two different numbers. We prove this intuition.

Theorem 1 (Uniqueness of Limit) *If a sequence is convergent, its limit is unique.*

So far, we have studied the convergence of sequences in general. Now we pay attention to a special kind of sequence, and examine its convergence property.

Definition 4 *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence.*

1. $\{a_n\}_{n=1}^{\infty}$ *is bounded from above if there exists* m *such that* $a_n \leq m$ *for all* n.
2. $\{a_n\}_{n=1}^{\infty}$ *is bounded from below if there exists* m *such that* $a_n \geq m$ *for all* n.
3. $\{a_n\}_{n=1}^{\infty}$ *is bounded if there exists* m *such that* $|a_n| \leq m$ *for all* n.

Having known the definition of bounded sets, this definition should not be too difficult to understood. The next important theorem states that all convergent sequences are bounded. That is, we can find a closed interval $[-m, m]$ that contains every term in the sequence.

Theorem 2 (Bounded Sequence) *Every convergent sequence is bounded.*

However, the converse is not true; i.e., bounded sequences are not necessarily convergent. Finally, we look at the useful properties of sequences. It turns out that sequences behave extremely well with respect to the operations of addition, multiplication, and division.

Theorem 3 (Algebraic Operations of Limit) *Let* $\lim a_n = x$, *and* $\lim b_n = y$.

1. $\lim ca_n = cx$ *for all* $c \in \mathbf{R}$.
2. $\lim(a_n + b_n) = x + y$.
3. $\lim(a_n b_n) = xy$.
4. $\lim \left(\frac{a_n}{b_n}\right) = \frac{x}{y}$ *if* $b_n \neq 0$ *for all* $n \in \mathbf{N}$ *and* $b \neq 0$.

Sequences also work well with the order.

Theorem 4 (Order of Limit) *Let* $\lim a_n = x$, *and* $\lim b_n = y$.

1. *If* $a_n \geq 0$ *for all* $n \in \mathbf{N}$, *then* $x \geq 0$.
2. *If* $a_n \leq b_n$ *for all* $n \in \mathbf{N}$, *then* $x \leq y$.
3. *If* $c \leq a_n$ *for some* $c \in \mathbf{R}$, *then* $c \leq x$.
4. *If* $a_n \leq c$ *for some* $c \in \mathbf{R}$, *then* $x \leq c$.

2 Cauchy Sequences

The following concept is very similar to the convergence of sequences given above,

Definition 5 *A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|a_n - a_m| < \epsilon$ for any $m, n \geq N$.*

Let's compare this definition with that of convergent sequences. The definition of a convergent sequence says that given an arbitrarily small positive number ϵ, it is possible to find a point in the sequence after which all the terms of sequences are close to the limit a within the range of ϵ. In contrast, the definition of a Cauchy sequence says that it is possible to find a point in the sequence after which all the terms are closer to each other within the range of ϵ. Intuitively, those two things are related to each other. In fact, we will eventually prove that they are equivalent.

Theorem 5 (Cauchy Sequences) *Two important theorems:*

1. *Every convergent sequence is a Cauchy sequence.*
2. *Every Cauchy sequence is bounded.*

Once you get far enough in a Cauchy sequence, you might suspect that its terms will start piling up around a certain point because they get closer and closer to each other. We call such a point an accumulation point (or limit point). We first consider an accumulation point of a set,

Definition 6 *x is an accumulation point of a set Q if every neighborhood of x contains infinitely many points of Q that are different from x.*

Then, we can similarly define an accumulation point of a sequence by considering its range as an infinite set. That is, x is a limit point of $\{a_n\}_{n=1}^{\infty}$ if for every $\epsilon > 0$ there are infinite number of terms a_n such that $|a_n - x| < \epsilon$. The following theorem makes this connection more explicit.

Theorem 6 (Accumulation Point) *x is an accumulation point of a set X if and only if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} a_n = x$ and $a_n \in X$ and $a_n \neq x$ for all $n \in \mathbf{N}$.*

Examples are always helpful to understand the key points about a new concept.

Example 3 *Three examples:*

1. *No finite set has an accumulation point.*
2. *The sequence $\{1, 1, 1, \ldots\}$ has 1 as the limit, but it has no accumulation point.*
3. *Every real number is an accumulation point of the set of rational numbers.*

The first example illustrates that every neighborhood of an accumulation point has to contain infinitely many points that belong to the set. The second example shows that the limit is not necessarily an accumulation point. The third example shows that an accumulation point itself may not belong to the set. The following theorem gives us the condition under which an accumulation point exists.

Theorem 7 (Bolzano-Weierstrass) *Every bounded infinite set has at least one accumulation point.*

Theorem 8 (Equivalence of Cauchy and Convergent Sequences) *Every Cauchy sequence is convergent.*

That is, a sequence is convergent if and only if it is a Cauchy sequence.

3 Subsequences and Monotone Sequences

As the final topic on sequences, we study two special kinds of sequences. The first is a monotone sequence.

Definition 7 *A sequence is monotone if it is either increasing or decreasing.*

1. *$\{a_n\}_{n=1}^{\infty}$ is increasing (decreasing) if $a_{n+1} \geq a_n$ ($a_{n+1} \leq a_n$) for all n.*
2. *$\{a_n\}_{n=1}^{\infty}$ is strictly increasing (strictly decreasing) if $a_{n+1} > a_n$ ($a_{n+1} < a_n$) for all n.*

Without much hard thinking, we can come up with a monotone sequence that is not convergent. Earlier, we also saw that although convergent sequences are bounded, the converse is not necessarily true. Here, we prove that if a bounded sequence is monotone, then it is convergent. Moreover, a monotone sequence converges only when it is bounded.

Theorem 9 (Monotone Convergence) *A monotone sequence is convergent if and only if it is bounded.*

Example 4 *Consider a sequence defined recursively, $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + \sqrt{a_{n-1}}}$ for $n = 2, 3, \ldots$. Is this sequence convergent? If so, what is the limit?*

Next, we consider a subsequence of a sequence. Loosely speaking, a subsequence is a remaining sequence after deleting some terms of the original sequence.

Definition 8 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then, $\{a_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{a_n\}_{n=1}^{\infty}$.*

Note that the original sequence itself is a subsequence.

Example 5 *Consider a sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. Find $\{n_k\}_{k=1}^{\infty}$ for the following subsequences:*

1. $\left\{\frac{1}{k}\right\}_{k=1}^{\infty}$
2. $\left\{\frac{1}{2k}\right\}_{k=1}^{\infty}$
3. $\left\{\frac{1}{k^2}\right\}_{k=1}^{\infty}$

All of the three subsequences in the above example converge to the same limit, namely 0, as the original sequence. This property holds in general.

Theorem 10 (Convergence of Subsequences) *A sequence converges if and only if all of its subsequences converge, and they all converge to the same limit.*

Finally, we prove the theorem which places Theorem 7 in the context of sequences.

Theorem 11 (Modified Bolzano-Weierstrass) *Every bounded sequence contains a convergent subsequence.*

We leave the proof of this theorem as an exercise, which is very similar to the proof of Theorem 7 and applies the "divide-and-conquer" strategy to the interval $[-m, m]$ where $m \geq |a_n|$ for all n. This concludes the chapter on sequences.

4 Limit Superior, Limit Inferior, and Sequences of Sets

We have studied convergent sequences, but we do not know much about divergent sequences. We can study the latter by introducing another concept of limits of sequences.

Definition 9 (Limit Superior and Inferior) *Let $\{a_n\}_{n=1}^\infty$ be a sequence. We define the limit superior and limit inferior of the sequence as follows,*

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{m\geq n} a_m, \quad \liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{m\geq n} a_m.$$

To understand this definition, consider a sequence $b_n = \sup\{a_m : m \geq n\}$. Then, $\lim_{n\to\infty} b_n = \inf_{n\geq 1}\sup_{m\geq n} a_m$ because b_n is a monotonically decreasing sequence. Therefore, we can think of lim sup and lim inf as limiting bounds of a sequence. This also means that lim sup and lim inf *always exists* if we allow for the possibility of ∞ and $-\infty$. For example, if $\sup\{a_m : m \geq n\}$ is not bounded, $\limsup_{n\to\infty} a_n = -\infty$, whereas if $\inf\{a_m : m \geq n\}$ is not bounded, then $\liminf_{n\to\infty} a_n = \infty$. The following examples should give you a good understanding of these concepts,

Example 6 *Find the infinimum, suprimum, limit, limit inferior, and limit superior of the following sequences if they exist.*

1. $\{a_n\}_{n=1}^\infty$ *where* $a_n = 1/n$.
2. $\{a_n\}_{n=1}^\infty$ *where* $a_n = (-1)^n$.
3. $\{a_n\}_{n=1}^\infty$ *where* $a_n = n(-1)^n$.

As we might expect, the limit superior is greater than or equal to the limit inferior, and they equal to each other if and only if the sequence is convergent.

Theorem 12 (Properties of Limit Superior and Inferior) *Let $\{a_n\}_{n=1}^\infty$ be a sequence. Then,*

1. $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.
2. $\lim a_n = x$ *if and only if* $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = x$.

These concepts and properties extend directly to a sequence of sets, $\{A_n\}_{n=1}^\infty$ where each A_n is a set rather than a real number, and its limit.

Definition 10 (Limit Superior and Inferior of a Sequence of Sets) *Let $\{A_n\}_{n=1}^\infty$ be a sequence of sets. Then, the limit superior and inferior of the sequence can be defined as,*

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m, \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^\infty \bigcap_{m=n}^\infty A_m.$$

These definitions are completely parallel with Definition 9 if we consider the supremum (infinimum) of a sequence of sets as the union (intersection) of sets, i.e., the smallest set containing all sets (the largest set contained in all sets). The same properties of limit superior and inferior hold for a sequence of sets.

Theorem 13 (Properties of Limit Superior and Inferior for Sequence of Sets) *Let $\{A_n\}_{n=1}^\infty$ be a sequence of sets. Then,*

1. $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.
2. *We say $\{A_n\}_{n=1}^\infty$ has a limit X and write* $\lim A_n = X$ *if and only if* $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = X$.